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A STUDY OF SOME GENERALIZATIONS OF PARACOMPACT AND METRIZABLE SPACES

by

© CHANDRA MOHAN PAREEK

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "A STUDY OF SOME GENERALIZATIONS OF PARA-COMPACT AND METRIZABLE SPACES", submitted by CHANDRA MOHAN PAREEK in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

One of the long outstanding open questions in point set topology asks whether or not each normal Moore space is metrizable. Our effort in this thesis is directed towards the study of intrinsic properties which characterize Moore spaces and the study of some related generalizations of metric spaces.

We begin in Chapter 1 with the investigation of a class of spaces introduced by Aragel'skii, called p-spaces. Various spaces related to p-spaces are also discussed.

Chapter 2 contains definitions, notations and some of the known results concerning networks for topological spaces, and spaces with G_δ-diagonal that are used in the following chapters. We believe that the notion of almost base and the characterization of spaces with G_δ-diagonal is new.

In Chapter 3 we study σ-paracompact spaces and their relation with some other well known topological spaces.

In Chapter 4 we give various characterizations of semi-metric spaces. The following is of special interest: A regular space is semi-metrizable iff it is first countable and has a σ-cushioned paired network.

In Chapter 5 we give various characterizations of quasi-metrizable spaces and a necessary and sufficient condition for a space to have a conjugate strong quasi-metric.

In Chapter 6 we give an intrinsic characterization of Moore spaces and show that every completely regular conjugate strong quasi-metric space is a Moore space.

In Chapter 7 we define the notions of weakly fundamental sequences of coverings and weak k-refining sequences of coverings, and we give a characterization of Nagata spaces in terms of weakly fundamental sequences of coverings, weak k-refining sequences of coverings and certain almost bases. Finally, we end this chapter, and the thesis, with some metrization theorems for the various classes of spaces we have studied here.

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TABLE OF CONTENTS

INTRODUCTION

Perhaps the most tantalizing open question remaining today in point set topology is the problem of determining whether each normal Moore space is metrizable. Our efforts in this thesis are directed towards the study of some intrinsic properties which characterize Moore spaces and the study of some related generalizations of metric spaces.

A Moore space is a topological space which satisfies Axiom 0 and the first three parts of Axiom 1 of [40]. In modern terminology a Moore space is a regular developable space. It is well known that every Moore space is semi-metrizable. On the other hand, Ceder [16] has shown that every Nagata space is semi-metrizable. Also, quasi-metric spaces have a natural relation with Moore spaces. We will show that a completely regular Moore space is a σ-paracompact, p-space with G -diagonal. Therefore, as promised in the previous paragraph, our subject of investigation becomes σ-paracompact spaces, p-spaces, spaces with G_δ-diagonals, semi-metric spaces, Nagata spaces, quasi-metric spaces and metrizable spaces.

In Chapter 1, we investigate p-spaces, strict p-spaces as defined by Arhangel'skii [7]; M-spaces, P-spaces by Morita [41], and the Δ-spaces, wΔ-spaces introduced by Borges [13]. The notable results in this chapter are characterizations of p-spaces without reference to compactification in Theorem 1.1.1 and the result that every wΔ-space is a P-space.

In Chapter 2, we define the notions of (i) almost base and (ii) σ-cushioned paired network for a topological space. We study

various properties of spaces with σ-cushioned paired networks. We also show that a topological space has G_δ-diagonal iff it is an open T_1-image of a metric space.

The main results of Chapter 3 are the following:

(a) Every F_σ-screenable space is σ-paracompact.

(b) X is F_σ-screenable iff every open cover has a σ-locally finite closed refinement.

(c) F_σ-screenable spaces are invariant under perfect maps in both directions.

(d) For locally compact Hausdorff spaces, X is F_σ-screenable iff X is a countable union of closed paracompact subspaces.

(e) If X is metacompact and every closed set is G_δ then X is F_σ-screenable.

(f) In countably metacompact spaces, X is screenable iff X is σ-fully normal.

(g) In σ-paracompact spaces, X is compact iff countably compact.

(h) A metacompact completely regular space need not be σ-paracompact. This answers a question of Arhangelskii [7].

In Chapter 4, we give several necessary and sufficient conditions for semi-metrizability. In particular, we show that a regular space is semi-metrizable iff it is first countable and has a σ-cushioned paired network.

In Chapter 5, we give various necessary and sufficient conditions for quasi-metrizability of topological spaces. We also give

a necessary and sufficient condition for a topological space to have a conjugate strong quasi-metric. Unfortunately, we could not find a necessary and sufficient condition that is "purely" topological.

In Chapter 6, we have obtained the following results:

(a) A completely regular space X is a Moore space iff X is σ-paracompact and the diagonal of X is a closed G_δ set in $X \times BX$, where BX is a Hausdorff compactification of X.

(b) A completely regular space X is a Moore space iff X is a σ-paracompact, p-space with G_δ-diagonal.

(c) A completely regular space X is a Moore space iff it is a p-space and has a σ-cushioned paried network. This answeres the question in [15].

(d) Every conjugate strong quasi-metric space is developable.

In Chapter 7, we show that the notions of weakly fundamental sequence of coverings and weak k-refining coverings are equivalent. A T_1-space is Nagata iff it has a weakly fundamental sequence of coverings or a weak k-refining sequence of coverings. We also show that for a regular space X to be Nagata it is necessary and sufficient that it is first countable and has a σ-cushioned paired almost base. Finally, we give some metrization theorems. The Theorem 7.3.3 is of interest as it answers the question raised by McAuley [35], "Is it possible to partition Bing's Theorem 4 of [11] into three or more parts which begins with a condition for a topological space and which ends with a condition for metrizability of the space, but with necessary and

sufficient conditons somewhere between these extremes for semi-metric and Moore Spaces?"

The following notation and terminology will be used throughout this thesis.

1. If $\underset{\sim}{A}$ is a collection of sets, then $\cup(A \mid A \in \underset{\sim}{A})$ is the set of all x such that $x \in A$ for some $A \in \underset{\sim}{A}$; sometimes, $\cup(A \mid A \in \underset{\sim}{A})$ will be denoted by $A^{\#}$.

2. If $\{\underset{\sim}{A}_\alpha \mid \alpha \in \Lambda\}$ is a family of sets such that each member is a family of sets, then $\{\underset{\sim}{A}_\alpha \mid \alpha \in \Lambda\} = \bigcup_{\alpha \in \Lambda} \{A \mid A \in \underset{\sim}{A}_\alpha\}$ where $\underset{\sim}{A}_\alpha = \{A \mid A \in \underset{\sim}{A}_\alpha\}$ for each $\alpha \in \Lambda$.

3. Unless explicitly stated otherwise, the letters i, j, k, m, n will denote variables whose values are natural numbers.

4. All other terms not defined here in are used as in [19] and [28] with the exception that regular and normal spaces are assumed T_1 .

CHAPTER I

p-SPACES

The concept of p-space is quite recent. It was introduced by Arhangel'skii [7]. The importance of this notion lies in the fact that a paracompact Hausdorff space admits a perfect mapping onto a metric space iff it is a p-space. Hence, in particular, it follows that the product of a countable set of paracompact Hausdorff p-spaces is a paracompact Hausdorff p-space. Furthermore, the closed image of a metrizable space is metrizable iff it is a p-space.

The definition of p-space given by Arhangel'skii [7] involves compactification of the space. In view of the above interesting properties of p-spaces Alexandroff [3] suggested a problem of finding a direct intrinsic definition (without appeal to compactification).

The investigations of this chapter are motivated by the above problem.

<u>Definition 1.1.1</u> A completely regular space X is called a <u>p-space</u> iff there is a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X in any one (hence in all) of its Hausdorff compactifications such that $\bigcap_{i=1}^{\infty} st(x,\underset{\sim}{V}_i) \subset X$, for all $x \in X$.

<u>Definition 1.1.2</u> Let $\{A_s \mid s \in S\}$ be a family of subsets of a set X and $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ be a countable family of covers of X . Then, we say that the family $\{A_s \mid s \in S\}$ has sets which are <u>base point small</u> relative

to $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ iff there exists $x_o \in X$ such that for each i, there is $s_i \in S$ and $V^i \in \underset{\sim}{V}_i$ for which $x_o \in V^i$ and $A_{s_i} \subset V^i$.

Theorem 1.1.1 A completely regular space X is a p-space iff there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X such that for every family of closed sets $\{F_s \mid s \in S\}$ which has the finite intersection property and contains sets which are base point small relative to $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ the inequality $\cap(F_s \mid s \in S) \neq \phi$ holds.

Proof. Let us suppose that there exists in X a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X which has the required property. Let $\underset{\sim}{V}_i = \{V_s^i \mid s \in S_i\}$ for $i = 1,2,\ldots,$ and let W_s^i denote an open set in βX (the Stone Čech Compactification of X) such that $V_s^i = W_s^i \cap X$ for $s \in S_i$ and $i = 1,2,\ldots$. Evidently, $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ where $\underset{\sim}{W}_i = \{W_s^i \mid s \in S_i\}$ is a countable family of open covers of X in βX for each i. We now show that $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{W}_i) \subset X$ for all $x \in X$.

Let $y \in \bigcap_{i=1}^{\infty} St(x,\underset{\sim}{W}_i)$, and let $\underset{\sim}{B}(y)$ be the family of all its neighborhoods in βX. The family $\{(Cl_{\beta X}B) \cap X \mid B \in \underset{\sim}{B}(y)\}$ consists of closed subsets of the space X and has the finite intersection property. Also, for each i there exists s_i such that x, y is in $W_{s_i}^i$. By the regularity of βX there is $B \in \underset{\sim}{B}(y)$ depending on i such that $y \in B$ and $Cl_{\beta X}B \subset W_{s_i}^i$. This implies that the family $\{(Cl_{\beta X}B) \cap X \mid B \in \underset{\sim}{B}(y)\}$ contains sets which are base point small relative to $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$, the base point being x. Therefore by the hypothesis $\cap(X \cap (Cl_{\beta X}B) \mid B \in \underset{\sim}{B}(y)) = X \cap (\cap(Cl_{\beta X}B \mid B \in \underset{\sim}{B}(y))) \neq \phi$.

But $\cap(\mathrm{Cl}_{\beta X}B \mid B \in \underset{\sim}{B}(y)) = y$, hence $y \in X$. Since y is an arbitrary member of $\bigcap_{i=1}^{\infty} \mathrm{St}(x,\underset{\sim}{V}_i)$, consequently $\bigcap_{i=1}^{\infty} \mathrm{St}(x,\underset{\sim}{V}_i) \subset X$.

Conversely, let us assume that X is a p-space, i.e., there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X in βX such that for each $x \in X$ we have $\bigcap_{i=1}^{\infty} \mathrm{St}(x,\underset{\sim}{V}_i) \subset X$. For each $x \in X$ and $i = 1,2,\ldots,$ let W_x^i be an open set in βX such that $x \in W_x^i \subset \mathrm{Cl}_{\beta X}W_x^i \subset V$ for some $V \in \underset{\sim}{V}_i$. We shall show that the countable family $\{\underset{\sim}{U}_i\}_{i=1}^{\infty}$ of open covers of the space X , where $\underset{\sim}{U}_i = \{X \cap W_x^i \mid x \in X\}$ has the required property.

Let $\{F_s \mid s \in S\}$ be a family of closed subsets of X which has the finite intersection property and contains sets which are base point small relative to $\{\underset{\sim}{U}_i\}_{i=1}^{\infty}$. The family $\{\mathrm{Cl}_{\beta X}F_s \mid s \in S\}$ has the finite intersection property and consists of closed subsets of βX . Therefore, by the compactness of βX , $\cap(\mathrm{Cl}_{\beta X}F_s \mid s \in S) \neq \phi$. Suppose $x \in \cap(\mathrm{Cl}_{\beta X}F_s \mid s \in S)$. Since $F_s = X \cap (\mathrm{Cl}_{\beta X}F_s)$, in order that $x \in \cap(F_s \mid s \in S)$, it is enough to show that $x \in X$.

Because $\{F_s \mid s \in S\}$ has sets which are base point small relative to $\{\underset{\sim}{U}_i\}_{i=1}^{\infty}$, there exists $x_o \in X$ such that for each i, one can choose $s_i \in S$ and $U^i \in \underset{\sim}{U}_i$ such that $F_{s_i} \subset U^i$ and $x_o \in U^i$. Since $x \in \mathrm{Cl}_{\beta X}F_{s_i} \subset \mathrm{Cl}_{\beta X}U^i \subset \mathrm{Cl}_{\beta X}W_{x_1}^i \subset \mathrm{St}(x_o,\underset{\sim}{V}_i)$, it follows that $x \in \mathrm{St}(x_o,\underset{\sim}{V}_i)$ for all i ; but, by the hypothesis $\bigcap_{i=1}^{\infty} \mathrm{St}(x_o,\underset{\sim}{V}_i) \subseteq X$. Consequently, $x \in X$. Hence the theorem is proved.

<u>Corollary 1.1.1.A</u> A completely regular space X is a p-space iff there is a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying:

(a) for each $x \in X$ and any sequence $\{V^i\}_{i=1}^{\infty}$ where $V^i \in \underset{\sim}{V}_i$ and $x \in V^i$ for each i, $A_x = \bigcap_{i=1}^{\infty} \overline{V}^i$ is compact; and

(b) the family $\{\overline{V}^i\}_{i=1}^{\infty}$ is the "base" for the open sets containing A_x.

The proof follows immediately from Theorem 1.1.1.

<u>Remark 1.1.1</u> If a completely regular space X is a p-space, then by Theorem 1.1.1, there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X such that for every family of closed subsets $\{F_s \mid s \in S\}$ of X which has the finite intersection property and contains sets which are base point small relative to $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ has non-empty intersection. Now, it is easy to show that the countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X is pointwise finitely additive; i.e., for each $x \in X$ and each $i = 1,2,\ldots$ and finitely many members $V_1^i, \ldots, V_{n_i}^i$ containing x, the $\bigcup_{j=1}^{n_i} V_j^i$ is an element of $\underset{\sim}{V}_i$.

<u>Theorem 1.1.2</u> Let X be a completely regular space and βX be the Stone Čech compactification of X. Then, X is a p-space iff there exists a sequence $\{G_i\}_{i=1}^{\infty}$ of open sets in $X \times \beta X$ such that $\Delta_X \subset \bigcap_{i=1}^{\infty} G_i \subset X \times X$, where $\Delta_X = \{(x,x) \mid x \in X\}$.

<u>Proof</u>. Let X be a p-space. Then there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X in βX such that $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i) \subseteq X$ for

all $x \in X$. Let us define $G_i = \cup(W \times V \mid W = V \cap X$ and $V \in \underset{\sim}{V}_i)$ for each i . Evidently, for each i , G_i is open in $X \times \beta X$ and contains Δ_X . We need only show that $\bigcap_{i=1}^{\infty} G_i \subset X \times X$. If $(x,y) \in \bigcap_{i=1}^{\infty} G_i$ then $y \in \bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i)$. Consequently, by the hypothesis, $y \in X$. Hence $\bigcap_{i=1}^{\infty} G_i \subset X \times X$.

Conversely, let us assume that $\Delta_X \subset \bigcap_{i=1}^{\infty} G_i \subset X \times X$, where G_i is open in $X \times \beta X$ for each i . For each $x \in X$ and $i = 1,2,\ldots$, let us choose an open neighborhood V_x^i of x in βX such that $(V_x^i \cap X) \times V_x^i \subset G_i$. We shall show that the countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$, where $\underset{\sim}{V}_i = \{V_x^i \mid x \in X\}$ for $i = 1,2,\ldots$, of open covers of X in βX has the required property, i.e., $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i) \subset X$ for all $x \in X$. Let $y \in \bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i)$, where x is any point of X . Then $(x,y) \in \bigcap_{i=1}^{\infty} G_i \subset X \times X$ implies $y \in X$. Hence the theorem is proved.

Arhangel'skii [7] has also defined the notion of a strict p-space, a stronger notion than that of p-space. Noticing that the definition of strict p-spaces again requires use of a compactification of the spaces involved, we now extend our investigation to look for an internal characterization of these spaces. One of our characterizations, the third, has also been given recently by Burke and Stoltenberg [12].

<u>Definition 1.2.1</u> A topological space X is called a <u>strict p-space</u> iff there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X in βX (where βX is the Stone Čech Compcatification of X) satisfying the following conditions:

(a) for each $x \in X$, $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i) \subset X$;

(b) for each $x \in X$ and i there is $j > i$ such that $Cl_{\beta X}(St(x,\underset{\sim}{V}_j)) \subset St(x,\underset{\sim}{V}_i)$.

Definition 1.2.2 Let $\{A_s \mid s \in S\}$ be a family of subsets of X and $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ be a countable family of open covers of X . Then we say $\{A_s \mid s \in S\}$ has sets which are base point strictly small relative $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ iff there exists $x_o \in X$ such that for each i there is $s_i \in S$ for which $A_{s_i} \subset St(x_o,\underset{\sim}{V}_i)$.

Theorem 1.2.1 In a completely regular space X the following statements are equivalent:

(i) X is a strict p-space;

(ii) there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X such that

(a) for each $x \in X$ and i there is $j > i$ such that $Cl_X(St(x,\underset{\sim}{V}_j)) \subset St(x,\underset{\sim}{V}_i)$, and

(b) for any family of closed sets $\underset{\sim}{F} = \{F_s \mid s \in S\}$ of X with the finite intersection property and contains sets which are base point strictly small relative to $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$, the inequality $\cap(F_s \mid s \in S) \neq \phi$ holds;

(iii) there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X such that for each $x \in X$, $A_x = \bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i)$ is compact and for any open $U \supset A_x$ there is an i such that $A_x \subset St(x,\underset{\sim}{V}_i) \subset U$ (Burke and Stoltenberg [12]).

Proof. (i) $\Longrightarrow$ (ii) Let X be a strict p-space and βX be the

Stone Čech Compactification of X. Let $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ be a countable family of open covers of X in βX satisfying (a) $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{W}_i) \subset X$ for all $x \in X$, and (b) for each $x \in X$ and any positive integer n there is $n^1 > n$ such that $Cl_{\beta X}(x,\underset{\sim}{W}_{n^1})) \subset St(x,\underset{\sim}{W}_n)$. Let us define $\underset{\sim}{V}_i = \{W \cap X \mid W \in W_i\}$ for $i = 1,2,\ldots$. We shall show that the countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X has the required property. It is easy to see that condition (a) of (ii) is satisfied by the countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X. We now show that (b) is also satisfied.

Let us assume that $\{F_s \mid s \in S\}$ is a family of closed subsets of X which has the finite intersection property and contains sets which are base point strictly small relative to $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$. The family $\{Cl_{\beta X}F_s \mid s \in S\}$ has the finite intersection property and $Cl_{\beta X} F_s$ is closed in βX for each $s \in S$, and so, by the compactness of βX, there is $x \in \cap(Cl_{\beta X}F_s \mid s \in S)$. Now, it is enough to show that $x \in X$.

For each natural number i let us choose j such that

$$Cl_{\beta X}(St(x_o,\underset{\sim}{W}_j)) \subset St(x_o,\underset{\sim}{W}_i) \quad .$$

Then, clearly $Cl_X(St(x_o,\underset{\sim}{V}_j)) \subset Cl_{\beta X}(St(x_o,\underset{\sim}{W}_j))$. Now, choose $s_{i,j} \in S$ such that

$$F_{s_{i,j}} \subseteq St(x_o,\underset{\sim}{V}_j) \quad .$$

Then $x \in Cl_{\beta X}F_{s_{i,j}} \subset Cl_{\beta X}(St(x_o,\underset{\sim}{V}_j)) \subset Cl_{\beta X}(st(x_o,\underset{\sim}{W}_j)) \subset St(x_o,\underset{\sim}{W}_i)$

for all i, implies that $x \in \bigcap_{i=1}^{\infty} St(x_o, \underset{\sim}{W}_i)$; but $\bigcap_{i=1}^{\infty} St(x_o, \underset{\sim}{W}_i) \subset X$ implies $x \in X$.

(ii) ==> (iii) Let us suppose there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the conditions in (ii) . Then we may assume that $\underset{\sim}{V}_{i+1}$ refines $\underset{\sim}{V}_i$ for each i . Let $x \in X$ and $A_x = \bigcap_{i=1}^{\infty} St(x, \underset{\sim}{V}_i)$ and let $\{F_s \mid s \in S\}$ be a family of closed subsets of A_x which has the finite intersection property. It is easy to see that by (a) A_x is closed. Therefore $\{F_s \mid s \in S\}$ is a collection of closed subsets of X which has the finite intersection property and contains sets which are base point strictly small relative to $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$. Hence $\cap(F_s \mid s \in S) \neq \phi$. Consequently, by Theorem 1 on page 137 of Kelley [28], A_x is compact. Since x is an arbitrary point of X , for each $x \in X$ we have shown that $A_x = \bigcap_{i=1}^{\infty} St(x, \underset{\sim}{V}_i)$ is a compact subset of X . Finally, let U be any open set containing A_x and suppose $St(x, \underset{\sim}{V}_i) \not\subset U$ for each i . Then, for each i choose $x_i \in St(x, \underset{\sim}{V}_i) - U$. Let us define $F_i = \cup(\{x_j\} \mid j \geq i)$. Then clearly $\{Cl_X F_n\}_{n=1}^{\infty}$ is a family of closed sets with the finite intersection property. Furthemore $\{Cl_X F_n\}_{n=1}^{\infty}$ contains sets which are base point strictly small relative to $\{V_i\}_{i=1}^{\infty}$. Hence, by the hypothesis, there is $y \in \cap(Cl_X F_i \mid i = 1,2,\ldots)$. By construction $\cap(Cl_X F_i \mid i = 1,2,\ldots) \subset X - U$ so that $y \notin U$; but $y \in \bigcap_{i=1}^{\infty} St(x, \underset{\sim}{V}_i) \subset U$ (because for each i there is $j > i$ such that $Cl_X F_j \subset Cl_X(St(x, \underset{\sim}{V}_j)) \subset St(x, \underset{\sim}{V}_i))$, a contradiction. This implies that, for some i_o, $St(x, \underset{\sim}{V}_{i_o}) \subset U$.

(iii) ⟹ (i) Let $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ be a countable family of open covers of X satisfying the required properties. For each i, let us define $\underset{\sim}{W}_i = \{W \mid W \text{ open in } \beta X,\ W \cap X \in \underset{\sim}{V}_i\}$. We shall show that the countable family $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ of open covers of X in βX has the required property. First, recall the fact that if U is any open set in βX then $Cl_{\beta X}(U \cap X) = Cl_{\beta X}\, U$. Now, by the hypothesis we know that for $x \in X$, $A_x = \bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i)$ is compact and $\{St(x,\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ forms a base for the neighborhoods of A_x. We first show that $\{Cl_{\beta X}(St(x,\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ forms a base for the neighborhoods of A_x in βX. Let U be an open set containing A_x. Then by the regularity of βX there is an open neighborhood G of A_x such that $A_x \subset G \subset Cl_{\beta X} G \subset U$. Now $A_x \subset G \cap X$ and $G \cap X$ is open in X. So there is an i such that $A_x \subset St(x,\underset{\sim}{V}_i) \subset G \cap X$. Now, clearly $Cl_{\beta X}(St(x,\underset{\sim}{V}_i)) \subset Cl_{\beta X} G \subset U$. Also, now using the fact recalled above, $St(x,\underset{\sim}{W}_i) \subset Cl_{\beta X} St(x,\underset{\sim}{V}_i)$. Again using the regularity fo βX we have $A_x = \bigcap_{i=1}^{\infty} St(x,\underset{\sim}{W}_i) \subset X$ and $St(x,\underset{\sim}{W}_{n'}) \subset Cl_{\beta X} G \subset St(x,\underset{\sim}{W}_n)$ for $n' > n$. Hence X is a strict p-space.

We shall now broaden our investigation to consider the relationship between the properties so far studied and some obviously related concepts which are defined below.

<u>Definition 1.3.1</u> A topological space X is an <u>M-space</u> iff there exists a normal sequence $\{\underset{\sim}{U}_n\}_{n=1}^{\infty}$ of open covers of X satisfying the following condition: if $\{A_1, A_2, \ldots, A_n, \ldots\}$ is a sequence of subsets of X, with the finite intersection property, and if there exists $x_o \in X$ such that, for each $n = 1,2,\ldots$, there

exists $A_k \subset St(x_o, \underset{\sim}{U}_n)$, then $\bigcap_{n=1}^{\infty} \bar{A}_n \neq \phi$.

<u>Definition 1.3.2</u> A topological space X is a <u>Δ^*-space</u> iff there exists a countable family $\{\underset{\sim}{U}_n\}_{n=1}^{\infty}$ of open covers of X satisfying the following conditions:

(a) for each $x \in X$ and each $n = 1,2,\ldots,$

$$Cl_X(St(x,\underset{\sim}{U}_{n+1})) \subset St(x,\underset{\sim}{U}_n) ;$$

(b) if $\{A_1, A_2, \ldots, A_n, \ldots\}$ is a sequence of subsets of X , with the finite intersection property, and if there exists $x_o \in X$ such that for each $n = 1,2,\ldots$, there exists some

$$A_k \subset St(x_o,\underset{\sim}{U}_n) , \text{ then } \bigcap_{n=1}^{\infty} \bar{A}_n \neq \phi .$$

<u>Definition 1.3.3</u> (a) A topological space X is said to be a <u>$\omega\Delta$-space</u> iff there exists a countable family $\{\underset{\sim}{U}_i\}_{i=1}^{\infty}$ of open covers of X such that for each $x \in X$, if x_n is in $St(x_o,\underset{\sim}{U}_n)$ for $n = 1,2,\ldots$, then the sequence $\{x_n\}$ has a cluster point.

(b) A topological space X is said to be a <u>Δ-space</u> iff X is a $\omega\Delta$-space and the covers $\underset{\sim}{U}_n$ satisfying (a) can be so choosen that we can also have, for each $x \in X$ and each $n = 1,2,\ldots$, $Cl_X(St(x,\underset{\sim}{U}_{n+1})) \subseteq St(x,\underset{\sim}{U}_n)$.

<u>Definition 1.3.4</u> A topological space X is a <u>P-space</u> in the Morita sense iff for any set $\underset{\sim}{A}$ of indices and for any family $\{G(\alpha_1,\ldots,\alpha_i) \mid \alpha_1,\ldots,\alpha_i \in \underset{\sim}{A};\ i = 1,2,\ldots\}$ of open subsets of X satisfying the condition:

(a) $G(\alpha_1,\ldots,\alpha_i) \subset G(\alpha_1,\ldots,\alpha_{i+1})$ for $\alpha_1,\ldots,\alpha_{i+1} \in \underset{\sim}{A}$ and for $i = 1,2,\ldots$,

there exists a family $\{F(\alpha_1,\ldots,\alpha_i) \mid \alpha_1,\ldots,\alpha_i \in \underset{\sim}{A}$; for $i = 1,2,\ldots\}$ of closed subsets of X satisfying the conditions (b) $F(\alpha_1,\ldots,\alpha_n) \subset G(\alpha_1,\ldots,\alpha_n)$ for $\alpha_1,\ldots,\alpha_n \in \underset{\sim}{A}$, and (c) $X = \bigcup_{i=1}^{\infty} F(\alpha_1,\ldots,\alpha_i)$ for any sequence $\{\alpha_i\}$ such that $X = \bigcup_{i=1}^{\infty} G(\alpha_1,\ldots,\alpha_i)$.

Proposition 1.3.1 Let X be a topological space. Then,

(i) X is a M-space implies X is a Δ^*-space;

(ii) X is a Δ^*-space implies X is a Δ-space;

(iii) X is a Δ-space implies X is a $\omega\Delta$-space;

(iv) X is a $\omega\Delta$-space implies X is a P-space in the Morita sense.

Proof. (i) If $\{\underset{\sim}{U}_i\}_{i=1}^{\infty}$ is a normal sequence of covers of X , then for each i and $x \in X$ we have $St(St(x,\underset{\sim}{U}_{i+1})) \subseteq St(x,\underset{\sim}{U}_i)$, i.e., $Cl_X(St(x,\underset{\sim}{U}_{i+1})) \subset St(x,\underset{\sim}{U}_i)$. Now, by the definition, it is easy to see that every M-space is a Δ^*-space.

(ii) and (iii) are obvious.

(iv) Let $\{G(\alpha_1,\ldots,\alpha_i) \mid \alpha_1,\ldots,\alpha_i \in \underset{\sim}{A}$, $i = 1,2,\ldots\}$ be a family of open subsets of X satisfying condition (a) of definition 1.3.4. Since X is a $\omega\Delta$-space there exists a countable family $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the condition (a) of definition 1.3.3. Without loss of generality we may assume that $\underset{\sim}{W}_{i+1}$ refines $\underset{\sim}{W}_i$ for each i . Let us now define

$$F(\alpha_1,\ldots,\alpha_i) = X - St((X - G(\alpha_1,\ldots,\alpha_i)), \underset{\sim}{W}_i) \quad .$$

Obviously, $F(\alpha_1,\ldots,\alpha_i) \subset G(\alpha_1,\ldots,\alpha_i)$ for each i . If $X = \bigcup_{i=1}^{\infty} G(\alpha_1,\ldots,\alpha_i)$ we want to show that $X = \bigcup_{i=1}^{\infty} F(\alpha_1,\ldots,\alpha_i)$. Suppose $x_o \in X - \bigcup_{i=1}^{\infty} F({}_1,\ldots,\alpha_i)$. Since $X - \bigcup_{i=1}^{\infty} F(\alpha_1,\ldots,\alpha_i) = \bigcap_{i=1}^{\infty} St((X - G(\alpha_1,\ldots,\alpha_i)), \underset{\sim}{W}_i)$ we have $(X - G(\alpha_1,\ldots,\alpha_i)) \cap St(x_o,\underset{\sim}{W}_i) \neq \phi$ for all i . Let us choose $x_i \in (X - G(\alpha_1,\ldots,\alpha_i)) \cap St(x_o,\underset{\sim}{W}_i)$ for each i . By the hypothesis $\{x_i\}_{i=1}^{\infty}$ has a cluster point which belongs to $\bigcap_{i=1}^{\infty} (X - G(\alpha_1,\ldots,\alpha_i))$, contrary to the fact that $\bigcup_{i=1}^{\infty} G(\alpha_1,\ldots\alpha_i) = X$. Consequently, $X = \bigcup_{i=1}^{\infty} F(\alpha_1,\ldots,\alpha_i)$. Hence X is a P-space in the Morita sense.

Proposition 1.3.2 Every P-space in the Morita sense is a countably metacompact space.

Proof. Follows from Theorem 2 on page 162 of Hayashi [23].

Proposition 1.3.3 Every strict p-space is a wΔ-space.

Proof. Follows from Theorem 1.2.1.

Proposition 1.4.1 Let X be a completely regular space with the following properties:

(a) X has no infinite compact set;

(b) X has at least one point which is not G_δ .

Then X is not point countable type.

Proof. Suppose X is point countable type. Let $x_o \in X$ be a point which is not a G_δ point of X. Since X is point countable type there exists a compact set K of countable character containing x. According to the choice of x_o, K is not a finite set, i.e., K is infinite, which contradicts (a). Hence X is not point countable type.

Corollary 1.4.1A If X is a completely regular space satisfying conditions (a) and (b) of Proposition 1.4.1, then X is not a p-space (X is not a strict p-space).

Proof. Follows from the fact that every p-space is point countable type.

Proposition 1.4.2 If X is a completely regular p-space such that each point is a G_δ set, then X is first countable.

Proof. Follows by Remark 1.1.1, Arhangel'skii [5], and Aull [9].

Theorem 1.4.3 There exists a completely regular countably compact space which is not a p-space.

Proof. By Novàk [44] there exists a completely regular countably compact space X such that $N \subset X \subset \beta N$ (where N denotes the set of positive integers with the discrete topology and βN is the Stone Čech compactification of N), $|X| \leq 2^{\aleph_0}$ and every infinite

set which is closed in X has cardinality $2^{2^{\aleph_0}}$. The space X has no infinite compact set. Also, no point of X which is in $X - N$ is a G_δ set. Actually, since X is countably compact every G_δ point is first countable. We shall show that no point of $X - N$ has countable neighborhood base. Suppose some $x_o \in X - N$ has a countable neighborhood base. It is then easy to see that there exists a continuous function f on X with $f(x_o) = 0$ and $f(y) > 0$ for $y \in X - \{x_o\}$. Evidently, $g(y) = \sin \frac{1}{f(y)}$ has no continuous extension across x_o, but this is not possible. Now, by Corollary 1.4.1A, X is not a p-space.

<u>Remark 1.4.1</u> Theorem 1.4.3 shows that an M-space need not be a p-space.

CHAPTER II

ON NETWORKS FOR TOPOLOGICAL SPACES AND TOPOLOGICAL SPACES WITH G_δ-DIAGONAL

This chapter contains definitions, notations and some of the known results concerning networks for topological spaces and spaces with G_δ-diagonal that are used in the following chapters.

The concept of a network for a topological space, or what is called a point pseudobase by Michael [36], was first introduced by Arhangel'skii [8]. In this chapter we define the notion of almost base and σ-cushioned paired network. The former is a generalization of the notion of pseudobase which was introduced by Michael [36] and the later is a generalization of the notion of σ-cushioned paired base by Ceder [16]. Our σ-cushioned paired networks have been considered recently by Kofner [31], who calls them σ-biconservative paired networks.

In Section 1, we study a few simple properties of almost bases for a topological space and also properties of some special networks.

In Section 2, we recall many results known for spaces with G_δ-diagonal and characterize spaces which admit a G_δ-diagonal.

<u>Definition 2.1.1</u> A collection $\underset{\sim}{B}$ of subsets of a topological X

is called a <u>network</u> for X iff for each $x \in X$ and any open subset U of X containing x there is $B \in \underset{\sim}{B}$ such that $x \in B \subset U$.

<u>Remark 2.1.1</u> If $\underset{\sim}{B}$ is a base for the topology of X, then $\underset{\sim}{B}$ is a network for X, but the converse is not true.

<u>Remark 2.1.2</u> If $\underset{\sim}{B}$ is a base for the topology of X, then for each compact set C of X and any open set U containing C there is a finite collection $B_1,\ldots,B_n \in \underset{\sim}{B}$ such that $C \subset \bigcup_{i=1}^{n} B_i \subset U$. In general the above property is not possessed by a network for a topological space.

<u>Definition 2.1.2</u> A network $\underset{\sim}{B}$ for a topological space X is called an <u>almost base</u> iff for each compact set K of X and any open set U containing K there exists a finite collection $B_1,\ldots,B_n \in \underset{\sim}{B}$ such that $K \subset \bigcup_{i=1}^{n} B_i \subset U$.

Obviously, a base for a topological space X is an <u>almost base</u> for X and an almost base for X is a network. But it is easy to construct examples to show that a network for a topological space need not be an almost base and an almost base need not be a base.

<u>Remark 2.1.3</u> In a regular space the notions of locally finite network and locally finite almost base are the same. However, there are regular spaces with a σ-locally finite network and without a σ-locally finite almost base and also regular spaces with a σ-locally finite almost base and without a σ-locally finite base.

In general an almost base is not a continuous invariant. As a matter of fact the continuous image of a base need not be an almost base.

<u>Definition 2.1.3</u> A continuous mapping $f : X \to Y$ of X onto Y is called a <u>compact covering</u> iff every compact subset of Y is the image of a compact subset of X.

The proof of the following proposition is immediate.

<u>Proposition 2.1.1</u> Let X be a topological space and $\underset{\sim}{B}$ be an almost base for X and let $f : X \to Y$ be a compact covering. Then $f(\underset{\sim}{B}) = \{f(B) \mid B \in \underset{\sim}{B}\}$ is an almost base for Y.

<u>Remark 2.1.4</u> If X is a regular space and $\underset{\sim}{B}$ is a network for X, then $\underset{\sim}{B}^1$, the collection of closures of members of $\underset{\sim}{B}$, is also a network for X.

<u>Definition 2.1.4</u> Let $\underset{\sim}{P}$ be a collection of ordered pairs $P = (P_1, P_2)$ of subsets of X, with $P_1 \subset P_2$ for all $P \in \underset{\sim}{P}$. Then $\underset{\sim}{P}$ is called a <u>paired network</u> iff for each $x \in X$ and an arbitrary neighborhood U of x, there is a $P \in \underset{\sim}{P}$ such that $x \in P_1 \subset P_2 \subset U$. $\underset{\sim}{P}$ is called a <u>cushioned paired collection</u> in X iff for each $\underset{\sim}{P}^1 \subset \underset{\sim}{P}$, $\overline{\cup(P_1 \mid P \in \underset{\sim}{P}^1)} \subset \cup(P_2 \mid P \in \underset{\sim}{P}^1)$. Finally, $\underset{\sim}{P}$ is called a <u>σ-cushioned paired network</u> for X iff $\underset{\sim}{P}$ is a paired network for X which can be written as a countable union of cushioned paired collections.

The proof of the following proposition is easy.

Proposition 2.1.2 If a topological space X has a σ-cushioned paired network, then every closed subset of X is a G_δ set.

Proposition 2.1.3 Let X be a countable product of spaces X_n ; $n = 1,2,\ldots$. If each X_n has σ-cushioned paired network, then X has σ-cushioned paired network.

Proof. For each n , let $\underset{\sim}{Q}_n = \bigcup_{m=1}^{\infty} \underset{\sim}{P}_m^n$ be a σ-cushioned paired network for X_n . Without loss of generality, let us assume that $(X_n, X_n) \in \underset{\sim}{P}_m^n$ and $\underset{\sim}{P}_m^n \subset \underset{\sim}{P}_{m+1}^n$. Now let $X = \prod_{n=1}^{\infty} X_n$, and define

$$\underset{\sim}{P}_n = \{ \left(\prod_{i=1}^{n} P_n^{i,1}, \prod_{i=1}^{n} P_n^{i,2} \right) \mid P_n^i \in \underset{\sim}{P}_n^i \}$$

where

$$\prod_{i=1}^{n} P_n^{i,1} = \{x \in X \mid x_i \in P_n^{i,1} \text{ for } i \leqq n\} = \prod_{i=1}^{n} P_n^{i,1} \times \prod_{j>n} X_j ,$$

and

$$\prod_{i=1}^{n} P_n^{i,2} = \{x \in X \mid x_i \in P_n^{i,2} \text{ for } i \leqq n\} = \prod_{i=1}^{n} P_n^{i,2} \times \prod_{j>n} X_j ,$$

We will use the notation $P^n = (P_1^n, P_2^n)$ for an element of the collection $\underset{\sim}{P}^n$ thus defined. Letting $\underset{\sim}{P} = \bigcup_{n=1}^{\infty} \underset{\sim}{P}^n$ we claim that $\underset{\sim}{P}$ is a σ-cushioned paired network for X . That it is a paired network follows by the construction. We now show that $\underset{\sim}{P}_n$ is cushioned for each n .

Letting $\underset{\sim}{P}_n^1 \subset \underset{\sim}{P}_n$, we need to show that $\overline{\cup(P_1^n \mid P^n \in \underset{\sim}{P}_n^1)} \subset$ $\cup(P_2^n \mid P^n \in \underset{\sim}{P}_n^1)$. Suppose $y \notin \cup(P_2^n \mid P^n \in \underset{\sim}{P}_n^1)$. Then

$$\bigcap_{i=1}^{n} (\cup(X_1 \times \ldots \times (X_i - \overline{\cup(P_n^{i,1} \mid P^n \in P_n^1)}) \times X_{i+1} \times \ldots))$$

is a neighborhood of y which does not intersect $(P_1^n \mid P^n \in \underset{\sim}{P}_n^1)$;

i.e., $\overline{\cup(P_1^n \mid P^n \in \underset{\sim}{P}_n^1)} \subset \cup(P_2^n \mid P^n \in P_n^1)$. Hence the proposition is proved.

Corollary 2.1.3A If X is a T_2-space with σ-cushioned paired network, then $\Delta_X = \{(x,x) \mid x \in X\}$ is a G_δ set in X ; i.e., X has a G_δ-diagonal.

Proposition 2.1.4 If X is a topological space with σ-cushioned network, then X has σ-cushioned paired network $\underset{\sim}{P} = \bigcup_{i=1}^{\infty} \underset{\sim}{P}_i$ satisfying the following four conditions:

(i) $\underset{\sim}{P}_i$ is a cushioned paired covering of X for each i ;

(ii) $\underset{\sim}{P}_i \subset P_{i+1}$ for each i ;

(iii) $\underset{\sim}{P}_i$ is closed under intersections; i.e., if $\underset{\sim}{P}_i = \{(p_{\alpha 1}^i, P_{\alpha 2}^i) \mid \alpha \in \Lambda_i\}$ then for any subset $A_i \subset \Lambda_i$ the pair $(\bigcap_{\alpha \in A_i} P_{\alpha 1}^i, \bigcap_{\alpha \in A_i} P_{\alpha 2}^i)$ belongs to $\underset{\sim}{P}_i$ for each i .

(iv) for each $x \in X$ there is a collection $\{P_\alpha^i\}_{i=1}^\infty$ where $P^i \in \underset{\sim}{P}_i$ such that for any neighborhood U of x there is i with the property that $x \in P_{\alpha 1}^i \subset P_{\alpha 2}^i \subset U$.

Proof. Let $\underset{\sim}{B} = \bigcup_{j=1}^{\infty} \underset{\sim}{B}_j$ be a given σ-cushioned paired network for X. Without loss of generality we may assume that $\underset{\sim}{B}_1 = \{(X,X)\}$. Let us define $\underset{\sim}{C}_i = \bigcup_{j \leqq i} \underset{\sim}{B}_i$ and let $\underset{\sim}{P}_i = \{(\bigcap_{\alpha \in F} C^i_{\alpha 1}, \bigcap_{\alpha \in F} C^i_{\alpha 2}) \mid F \subset \wedge_i\}$ where $\underset{\sim}{C}_i = \{(C^i_{\alpha 1}, C^i_{\alpha 2}) \mid \alpha \in \wedge_i\}$. We do not consider a pair $(\bigcap_{\alpha \in F} C^i_{\alpha 1}, \bigcap_{\alpha \in F} C^i_{\alpha 2})$ in $\underset{\sim}{P}_i$ for which $\bigcap_{\alpha \in F} C^i_{\alpha 1} = \phi$. We claim that $\underset{\sim}{P} = \bigcup_{i=1}^{\infty} \underset{\sim}{P}_i$ satisfies all the four properties in the hypothesis. Let i be a fixed but arbitrary index and let $\wedge' \subset 2^{\wedge_i}$ where $2^{\wedge_i}$ denotes the collection of all non-empty subsets of $\wedge_i$. We now need to show that $\overline{\cup(\bigcap_{\alpha \in F} C^i_{\alpha 1} \mid F \in \wedge')} \subset \cup(\bigcap_{\alpha \in F} C^i_{\alpha 2} \mid F \in \wedge')$. Suppose $y \notin \cup(\bigcap_{\alpha \in F} C^i_{\alpha 2} \mid F \in \wedge')$. Then for each $F \in \wedge'$ there is α_F such that $y \notin C^i_{\alpha_F 2}$, i.e., $y \notin \cup(C^i_{\alpha_F 2} \mid F \in \wedge')$. Then $y \notin \overline{\cup(C^i_{\alpha_F 1} \mid F \in \wedge')}$ so that $y \in X - \overline{\cup(C^i_{\alpha_F 1} \mid F \in \wedge')}$. Now, let us assume that $(X - \overline{\cup(C^i_{\alpha_F 1} \mid F \in \wedge')}) \cap (\cup(\bigcap_{\alpha \in F} C^i_{\alpha 1} \mid F \in \wedge')) \neq \phi$, i.e., there is $z \in (X - \cup(C^i_{\alpha_F 1} \mid F \in \wedge')) \cap (\cup(\bigcap_{\alpha \in F} C^i_{\alpha 1} \mid F \in \wedge'))$. This implies that for some $F_o \in \wedge'$ we have $z \in (X - \cup(C^i_{\alpha_F 1} \mid F \in \wedge')) \cap (\bigcap_{\alpha \in F_o} C^i_{\alpha 1})$ which is impossible since there $z \notin C^i_{\alpha_{F_o} 1}$ and $z \in \bigcap_{\alpha \in F_o} C^i_{1}$ where $\alpha_{F_o} \in F_o$. Hence, for each i, $\underset{\sim}{P}_i$ is cushioned collection and it is a cover because the pair $(X,X) \in \underset{\sim}{P}_i$ for each i.

From the above argument and the construction of $\underset{\sim}{P}$, it is easy to see that $\underset{\sim}{P}$ satisfies (i), (ii) and (iii). We now need to show that (iv) is satisfied. For each $x \in X$ and i define $\wedge^i_x = \{\alpha \in \wedge_i \mid x \in C^i_{\alpha 1}\}$. Consider the pair $P^i_x = (\bigcap_{\alpha \in \wedge^i_x} C^i_{1}, \bigcap_{\alpha \in \wedge^i_x} C^i_{2})$

for each i . Now, if U is any neighborhood of x , then by the fact that $\underset{\sim}{B}$ is a paired network there is i such that $x \in B_1^i \subset B_2^i \subset U$. It is easy to see by construction that $x \in \bigcap_{\alpha\in\Lambda_x^i} C_{\alpha 1}^i \subset \bigcap_{\alpha\in\Lambda_x^i} C_{\alpha 2}^i \subset U$ as $\bigcap_{\alpha\in\Lambda_x^i} C_1^i \subset B_1^i$ and $\bigcap_{\alpha\in\Lambda_x} C_{\alpha 2}^i \subset B_2^i$. Hence the proposition is proved.

<u>Definition 2.1.5</u> A network $\underset{\sim}{B}$ for a topological space X is called <u>σ- closure preserving</u> (<u>σ-locally finite</u> or <u>σ-discrete</u>) provided $\underset{\sim}{B}$ can be written as a countable union of closure preserving (locally finite or discrete) collections.

<u>Proposition 2.1.5</u> If X is a topological space with a σ-closure preserving closed network, then X has a σ-closure preserving closed network $\underset{\sim}{P} = \bigcup_{i=1}^{\infty} \underset{\sim}{P}_i$ satisfying the following conditions:

(i) $\underset{\sim}{P}_i$ is a closure preserving closed covering of X for each i ;

(ii) $\underset{\sim}{P}_i \subset \underset{\sim}{P}_{i+1}$ for each i ;

(iii) $\underset{\sim}{P}_i$ is closed under intersection;

(iv) for each $x \in X$ there exists a collection $\{P_x^i\}_{i=1}^{\infty}$ such that for each i , $P_x^i \in \underset{\sim}{P}_i$ and for each neighborhood U of x there is i with the property that $x \in P_x^i \subset U$.

<u>Proof</u>. Similar to Proposition 2.1.4.

<u>Proposition 2.1.6</u> If X is a topological space with a σ-locally

finite closed network, then X has a σ-locally finite network $\underset{\sim}{P} = \bigcup_{i=1}^{\infty} \underset{\sim}{P}_i$ satisfying the following conditions:

(i) $\underset{\sim}{P}_i$ is a locally finite closed covering of X for each i ;

(ii) $\underset{\sim}{P}_i \subset \underset{\sim}{P}_{i+1}$ for each i ;

(iii) $\underset{\sim}{P}_i$ is closed under finite intersection;

(iv) for each $x \in X$ there exists a collection $\{P_x^i\}_{i=1}^{\infty}$ such that $P_x^i \in \underset{\sim}{P}_i$ for each i and for each neighborhood U of x there is P_x^i such that $x \in P_x^i \subset U$.

Proof. See K. Nagami [43].

Theorem 2.1.7 For a regular space X the following statements are equivalent:

(i) X has a σ-discrete network;

(ii) X has a σ-locally finite network;

(iii) X has a σ-closure preserving network.

Proof. See Siwiec and Nagata [51].

Proposition 2.1.8 Let $f : X \to Y$ be a continuous and closed mapping of a regular space X onto a T_1-space Y . Then

(i) Y has a σ-discrete network if X has a σ-discrete network;

(ii) Y has a σ-locally finite network if X has a σ-locally finite network;

(iii) Y has a σ-closure preserving network if X has a σ-closure preserving network;

(iv) Y has a σ-cushioned paired network if X has a σ-cushioned paired network.

Proof. Is obvious.

Definition 2.2.1 A topological space X is said to have a G_δ-diagonal iff $\Delta_X = \{(x,x) \mid x \in X\}$ is a G_δ set in $X \times X$.

Theorem 2.2.1 The diagonal Δ_X of a topological space X is a G_δ set in $X \times X$ iff there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X such that for each $x \in X$ we have $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i) = \{x\}$.

Proof. See Ceder [16].

Corollary 2.2.1A If a topological space X has a G_δ-diagonal, then X is a T_1-space.

We note the following simple properties of spaces with G_δ-diagonal.

Theorem 2.2.2 Let X be a topological space.

(a) If $X = \prod_{i=1}^{\infty} X_i$ where for each i, X_i has G_δ-diagonal, then X has a G_δ-diagonal.

(b) If X is a space with G_δ-diagonal, then every subset A of X has the property that Δ_A is a G_δ set in $A \times A$; i.e, A has G_δ-diagonal.

(c) If X is a space with G_δ-diagonal, then X has a G_δ-diagonal with respect to every stronger topology.

<u>Proposition 2.2.3</u> If a topological space $X = \bigcup_{i=1}^{\infty} X_i$ where for each i , X_i is a closed G_δ set in X and such that Δ_{X_i} is a G_δ set in $X_i \times X_i$, then X has a G_δ-diagonal.

<u>Proof</u>. By Theorem 2.2.1, for each i there exists a countable family $\{\underset{\sim}{V}_j^i\}_{j=1}^{\infty}$ of open covers of X_i such that for each $x \in X_i$ we have $\bigcap_{j=1}^{\infty} St(x,\underset{\sim}{V}_j^i) = \{x\}$. Let $X_i = \bigcap_{j=1}^{\infty} U_j^i$, where U_j^i is open in X . Without loss of genrality let us assume that $U_{j+1}^i \subset U_j^i$ for $j = 1,2,\ldots$. For a fixed i and j let us define $\underset{\sim}{W}_j^i$ the collection of open sets W in X such that $W \cap X_i$ is a member of $\underset{\sim}{V}_j^i$ and $W \subset U_j^i$. We now consider the sequence of covers $\{\underset{\sim}{W}_j^{i\#}\}$, where for each fixed i and arbitrary j , $\underset{\sim}{W}_j^{i\#} = \{(X - X_i)\} \cup \underset{\sim}{W}_j^i$. We shall show that for each $x \in X$, $\bigcap_{i,j=1}^{\infty} St(x,\underset{\sim}{W}_j^{i\#}) = \{x\}$. Let $x \in X$. Then $x \in X_i$ for some i , and by the construction $\bigcap_{j=1}^{\infty} St(x,\underset{\sim}{W}_j^{i\#}) = \{x\}$ as $X_i = \bigcap_{j=1}^{\infty} U_j^i$. Hence the proposition is proved.

<u>Definition 2.2.2</u> A mapping $f : X \to Y$ from a metric space X onto Y is said to be a T_1-mapping iff the pre-image of any two distinct

points of Y are at a positive distance.

<u>Theorem 2.2.4</u> A topological space Y has a G_δ-diagonal iff it is an open T_1-image of a metric space.

<u>Proof</u>. Let X be a metric space with metric ρ and f be an open T_1-mapping of X onto a topological space Y . We want to show that Y has a G_δ-diagonal. For each positive integer n let us define $\underset{\sim}{V}_n = \{S(x,n) \mid x \in X\}$ where $S(x,n)$ for each $x \in X$ and positive integer n denotes the sphere of radius $\frac{1}{n}$. Obviously, $\underset{\sim}{V}_n$ for each n is an open cover of X . Since f is onto and open, $\underset{\sim}{W}_n = f(\underset{\sim}{V}_n) = \{f(S(x,n)) \mid x \in X\}$ is an open cover of Y for each n . By Theorem 2.2.1, to show that Y has G_δ-diagonal, it is enough to show that $\bigcap_{n=1}^{\infty} St(y,\underset{\sim}{W}_n) = \{y\}$ for each $y \in Y$.

Suppose for some $y_o \in Y$ we have $y_1 \in \bigcap_{n=1}^{\infty} St(y_o,\underset{\sim}{W}_n)$ and $y_1 \neq y_o$. Since $y_1 \neq y_o$ and f is a T_1-mapping, we have $\rho(f^{-1}y_1, f^{-1}y_o) > 0$ i.e., there is an integer n_o such that $\rho(f^{-1}y_1, f^{-1}y_o) > \frac{1}{n_o}$. But this will imply that $f^{-1}y_o \cap St(f^{-1}y_o, \underset{\sim}{V}_{n_o}) = \phi$, i.e., $y_1 \notin St(y_o,\underset{\sim}{W}_n)$, which contradicts the fact that $y_1 \in \bigcap_{n=1}^{\infty} St(y_o,\underset{\sim}{W}_n)$. Hence Y has a G_δ-diagonal.

Conversely, suppose that Y has a G_δ-diagonal. Then, by Theorem 2.2.1 there exists a sequence $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of Y such that $\bigcap_{i=1}^{\infty} St(y,\underset{\sim}{V}_i) = \{y\}$ for each $y \in Y$. Without loss of generality we may assume that $\underset{\sim}{V}_{i+1}$ refines $\underset{\sim}{V}_i$ for each i . Let

$\underset{\sim}{V}_i = \{V^i_\alpha | \alpha \in \wedge_i\}$ for each i. Let $\wedge = \prod_{i=1}^{\infty} \wedge_i$ where each $\wedge_i$ carries the discrete topology and $\wedge$ carries the usual product topology. Each $\wedge_i$ has the discrete topology, i.e., for any $\alpha_1, \alpha_2 \in \wedge_i$ we can define a metric ρ_i such that $\rho_i(\alpha_1, \alpha_2) = 1$ if $\alpha_1 \neq \alpha_2$ and zero otherwise. Then if $(\alpha_1, \ldots, \alpha_n, \ldots)$ and $(\beta_1, \ldots, \beta_n, \ldots)$ are in $\wedge$, define $\rho((\alpha_1, \ldots, \alpha_n, \ldots), (\beta_1, \ldots, \beta_n, \ldots)) = [\sum_{n=1}^{\infty} \frac{1}{2} \rho^2_n(\alpha_n, \beta_n)]^{1/2}$. Let $\alpha = \{\alpha_i\}$ be a point of $\wedge$. If $\bigcap_{i=1}^{\infty} V^i{}_i \neq \phi$ and $x \in \bigcap_{i=1} V^i_\alpha$, then $x = \bigcap_{i=1}^{\infty} V^i_\alpha$ since $\bigcap_{i=1}^{\infty} St(x, \underset{\sim}{V}_i) = \{x\}$. In this case we put $f(\alpha) = x$. If $\bigcap_{i=1}^{\infty} V^i_\alpha = \phi$ then the map f is not defined.

Let $X \subset \wedge$ be the set of $\alpha \in \wedge$ such that f is defined. Since every subspace of a metric space is a metric space, X is a metric space. It remains to show that f is an open, T_1-mapping.

First, we show that f is onto. For if $y \in Y$, we can select $\alpha = \{\alpha_i\}$, $\alpha_i \in \wedge_i$ such that $y \in V^i_{\alpha_i}$ for $i = 1,2,\ldots,$ as $\underset{\sim}{V}_i$ for each i is an open cover of Y. But then $\bigcap_{i=1}^{\infty} V^i_{\alpha_i} = y$, so that $f(\alpha) = y$, where $\alpha \in X \subset \wedge$.

We now show that f is open. Let $\alpha^o \in X$ be an arbitrary point, say $\alpha^o = \{\alpha^o_i\}_{i=1}^{\infty}$. Then it is enough to show that for any neighborhood 0_{α^o} of α^o there is a neighborhood $0_{1\alpha^o}$ of α^o such that $0_{1\alpha^o} \subset 0_{\alpha^o}$ and $f0_{1\alpha^o}$ is open. Let

$0^{(n_1,\ldots,n_k)}_{\alpha^o} = \{\alpha = \{\alpha_n\} \in \wedge \,|\, \alpha_{n_i} = \alpha^o_{n_i}, i = 1,\ldots,k\}$ be a basic

neighborhood of α^o contained in 0_{α^o} and let

$0_{1\alpha^o} = \{\alpha = \{\alpha_n\} \in X \mid \alpha_{n_i} = \alpha^o_{n_i}$ for $i = 1,\ldots,k$, $\alpha_{n_o} = \alpha^o_{n_o}$ where

n_o is any index $\neq n_i\}$. Then $\alpha^o \in 0_{1\alpha^o} \subset 0^{(n_1,\ldots,n_k)}_{\alpha^o} \subset 0_{\alpha^o}$.

Now by the definition of f we have $f0_{1\alpha^o} = \bigcap_{i=o}^{k} V^{n_i}_{\alpha^o_{n_i}}$. Clearly

$f0_{1\alpha^o}$ is open as it is a finite intersection of open sets.

Finally, it remains to show that f is a T_1-mapping. Let $y_1 \neq y_2$ be two distinct points of Y . Then there is a k such that $y_2 \notin St(y_1,\underset{\sim}{V}_k)$. Therefore, if $\alpha = (\alpha_1,\ldots,\alpha_k,\ldots) \in f^{-1}y$, and $\beta = (\beta_1,\ldots,\beta_k,\ldots) \in f^{-1}y_2$ then $\alpha_k \neq \beta_k$ and $\rho((\alpha_1,\ldots,\alpha_k,\ldots), (\beta_1,\ldots,\beta_k,\ldots)) \geqq \frac{1}{k}$. Consequently $\rho(f^{-1}(x_1),f^{-1}(x_2)) \geqq \frac{1}{k}$. Hence the theorem is proved.

<u>Definition 2.2.3</u> A topological space X is said to have <u>$\bar{G}_\delta$-diagonal</u> iff there exists a sequence $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X such that for $x,y \in X$ there is n_x with the property that $y \notin Cl_X\, St(x,\underset{\sim}{V}_{n_x})$.

CHAPTER III

σ-PARACOMPACT AND F_σ-SCREENABLE SPACES

The concept of σ-paracompactness was introduced by Arhangel'skii [4], who obtained a few properties of σ-paracompact spaces, conjectured many and, remarked that it will be worth investigating further properties.

The study of the class of σ-paracompact spaces is the subject of Section 1. There we give some necessary conditions for a space to be σ-paracompact and study various operations.

In Section 2 we study F_σ-screenable spaces introduced by McAuley [34]. We give a characterization of F_σ-screenable spaces and as a corollary obtained that every F_σ-screenable space is σ-paracompact. We also study various operations on F_σ-screenable spaces.

In Section 3 we establish some interrelations between σ-paracompact and other well known topolgical spaces.

In Section 4 we give some examples.

<u>Definition 3.1.1</u> A topological space X is <u>σ-paracompact</u> iff for every open cover $\underset{\sim}{U}$ of X there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the condition: for any $x \in X$ there exists i such that $St(x,\underset{\sim}{V}_i) \subset U$ for some $U \in \underset{\sim}{U}$.

<u>Definition 3.1.2</u> A topological space X is <u>fully normal</u> iff every open cover of X has an open point star refinement; i.e., for each open cover $\underset{\sim}{U}$ of X, there is an open cover $\underset{\sim}{V}$ such that the cover $\{St(x,\underset{\sim}{V}) \mid x \in X\}$ refines $\underset{\sim}{V}$.

<u>Definition 3.1.3</u> An open cover $\underset{\sim}{U}$ of a topological space X is called a <u>σ-even</u> cover iff there exists a sequence of open neighborhoods $\{V_i\}_{i=1}^{\infty}$ of the diagonal in $X \times X$ such that for each $x \in X$ there is i for which $V_i[x] \subset U$ for some $U \in \underset{\sim}{U}$.

The definition of fully normal immediatly implies -

<u>Proposition 3.1.1</u> Every fully normal space is σ-paracompact.

<u>Proposition 3.1.2</u> If X is a σ-paracompact space, then every open cover is a σ-even cover.

<u>Proof</u>. Let $\underset{\sim}{U}$ be an open cover of a σ-paracompact space X. Then there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the condition that for any $x \in X$ there is i such that $St(x,\underset{\sim}{V}_i) \subset U$ for some $U \in \underset{\sim}{U}$. Let $G_i = \cup(V \times V \mid V \in \underset{\sim}{V}_i)$ for $i = 1,2,\ldots$. Obviously, $\{G_i\}_{i=1}^{\infty}$ is a countable sequence of neighborhoods of the diagonal in $X \times X$. We shall show that the countable sequence $\{G_i\}_{i=1}^{\infty}$ has the required property.

Let $x \in X$ and i be such that $St(x,\underset{\sim}{V}_i) \subset U_o$ for some $U_o \in \underset{\sim}{U}$. We shall show that $G_i[x] \subset St(x,\underset{\sim}{V}_i)$. For if $y \in G_i[x]$,

then $(x,y) \in G_i = \cup(V \times V \mid V \in \underset{\sim}{V}_i)$, i.e., x and y is in some $V \in \underset{\sim}{V}_i$ implies $y \in St(x,\underset{\sim}{V}_i)$. Since y is an arbitrary point of $G_i[x]$, implies $G_i[x] \subset St(x,\underset{\sim}{V}_i)$. But $St(x,\underset{\sim}{V}_i) \subset U_o$, so $G_i[x] \subset U_o$. Hence the proposition is proved.

<u>Proposition 3.1.3</u> If X is a σ-paracompact space, then every open cover of X has a σ-cushioned closed refinement.

<u>Proof</u>. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of a σ-paracompact space X. Then there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the condition that for $x \in X$ there is i such that $St(x,\underset{\sim}{V}_i) \subset U_s$ for some $s \in S$. Let us define $U_{s,i} = \{x \in X \mid St(x,V_i) \subset U_s\}$ for $s \in S$ and $i = 1,2,\ldots$. For each $s \in S$ and $i = 1,2,\ldots,$ the set $U_{s,i}$ is closed. Because if y is a limit point of the set $U_{s,i}$, then every member of $\underset{\sim}{V}_i$ which contains y will also contain some point x of $U_{s,i}$. But $St(x,\underset{\sim}{V}_i)$ for all $x \in U_{s,i}$ is contained in U_s, so that $St(y,\underset{\sim}{V}_i) \subset U_s$; i.e., $y \in U_{s,i}$. Consequently, $U_{s,i}$ is closed for each $s \in S$ and $i = 1,2,\ldots$. Let us denote by $\underset{\sim}{U}_i = \{U_{s,i} \mid s \in S\}$ for $i = 1,2,\ldots$.

It follows from the preceding paragraph that $\bigcup_{i=1}^{\infty} \underset{\sim}{U}_i$ is a closed refinement of $\underset{\sim}{U}$. We want to show that it is a σ-cushioned refinement. For this, we need only show that for each i and any subset S^1 of S we have

$$\overline{\cup(U_{s,i} \quad s \in S^1)} \subset \cup(U_s \mid s \in S^1) \quad .$$

Let y be a limit point of $\cup(U_{s,i} \mid s \in S^1)$. Then every member of $\underset{\sim}{V}_i$ which contains y has a non-empty intersection with $\cup(U_{s,i} \mid s \in S^1)$. This implies that $St(y,\underset{\sim}{V}_i) \subset \cup(U_s \mid s \in S^1)$ so $y \in \cup(U_s \mid s \in S^1)$. Since for each $s \in S$, $U_{s,i} \subset U_s$ we have $\overline{\cup(U_{s,i} \mid s \in S^1)} \subset \cup(U_s \mid s \in S^1)$ for each i and any subset S^1 of S . Consequently, $\bigcup_{i=1}^{\infty} \underset{\sim}{U}_i$ is a σ-cushioned closed refinement of $\underset{\sim}{U}$.

<u>Corollary 3.1.3A</u> If X is σ-paracompact, then for any well ordered monotone decreasing family $\{H_\alpha \mid \alpha \in \Lambda\}$ of closed sets with empty intersection, there is a monotone decreasing family of G_δ sets $\{G_\alpha \mid \alpha \in \Lambda\}$ such that

(a) $H_\alpha \subset G_\alpha$ for all $\alpha \in \Lambda$, and

(b) $\cap(G_\alpha \mid \alpha \in \Lambda) = \phi$.

<u>Proposition 3.1.4</u> Every σ-paracompact space X is countably meta-compact.

<u>Proof</u>. Let $\underset{\sim}{U} = \{U_i\}_{i=1}^{\infty}$ be a countable open cover of a σ-paracompact space X . Then there exists a countable family $\{\underset{\sim}{V}_j\}_{j=1}^{\infty}$ of open covers of X satisfying the condition that for any $x \in X$ there is j such that $St(x,\underset{\sim}{V}_j) \subset U_i$ for some i . Let us define $W_{ij} = \{x \in X \mid St(x,\underset{\sim}{V}_j) \subset U_i\}$ for each i and j . As in Proposition 3.1.3, W_{ij} is a closed subset of X for each i and j . Letting $\underset{\sim}{W} = \{W_{ij}\}_{i,j=1}^{\infty}$, it is clear that $\underset{\sim}{W}$ is a countable closed refinement of $\underset{\sim}{U}$ and $W_{ij} \subset U_i$ for each fixed i and arbitrary j . The family

$\underset{\sim}{W}$ is countable, so we can index by integers and say $\underset{\sim}{W} = \{W_k\}_{k=1}^{\infty}$. Now, $\underset{\sim}{W}$ being a refinement of $\underset{\sim}{U}$, for each k there exists an integer i_k such that $W_k \subset U_{i_k}$. Let us define $B_k = U_{i_k}$ and $\underset{\sim}{B} = \{B_k\}_{k=1}^{\infty}$. Then $\underset{\sim}{B} = \{B_k\}_{k=1}^{\infty}$ is an open cover of X such that $\underset{\sim}{W} = \{W_k\}_{k=1}^{\infty}$ is a closed refinement and $W_k \subset B_k$ for each k.

Let us define $R_1 = B_1$ and $R_n = B_n \cap (\bigcap_{i=1}^{n=1} (X - W_i))$ for $n > 1$. Clearly, for each n, R_n is an open set and $R_n \subset B_n$ for each n. We need only show that $\{R_n\}_{n=1}^{\infty}$ is a point finite cover of X. $\{R_n\}_{n=1}^{\infty}$ is a cover of X, for if $x \in X$ and $x \notin R_1$ then there is first n such that $x \in B_n$ and $x \notin B_m$ for $m < n$. Then, by the construction of R_n we have $x \in R_n$. Therefore $\{R_n\}_{n=1}^{\infty}$ is an open cover of X. $\{R_n\}_{n=1}^{\infty}$ is point finite because $\{W_k\}_{k=1}^{\infty}$ is a countable closed cover of X. Thus, we have shown that $\{R_n\}_{n=1}^{\infty}$ is a point finite open refinement of $\{B_k\}_{k=1}^{\infty}$ and therefore of $\underset{\sim}{U}$. Hence X is countably metacompact.

<u>Proposition 3.1.5</u> A topological space X is hereditarily σ-paracompact iff every open subspace X is σ-paracompact.

<u>Proof</u>. If X is hereditarily σ-paracompact, then every subspace of X is σ-paracompact, and hence every open subspace of X is σ-paracompact.

Conversely, let us assume that every open subset of X is σ-paracompact, and let A be any subset of X. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be a cover of A by sets open in A and let $\underset{\sim}{V} = \{V_s \mid s \in S\}$ be a

family of open sets in X such that $U_s = V_s \cap A$ for $s \in S$. Let $V^{\#} = \cup(V_s \mid s \in S)$. Obviously, $V^{\#}$ is an open subset of X, and so by the hypothesis $V^{\#}$ is σ-paracompact. Since $V^{\#}$ is σ-paracompact there exists a sequence $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of $V^{\#}$ satisfying the condition that for any $x \in V^{\#}$ there is i such that $St(x,\underset{\sim}{V}_i) \subset V_s$ for some $s \in S$. For each i let us define $\underset{\sim}{U}_i = \{V \cap A \mid V \in \underset{\sim}{V}_i\}$. Clearly, $\{\underset{\sim}{U}_i\}_{i=1}^{\infty}$ is a sequence of covers of A by open sets in A and if $x \in A$, then $x \in V^{\#}$ and so there is i such that $St(x,\underset{\sim}{V}_i) \subset V_s$ for some $s \in S$. Consequently, $St(x,\underset{\sim}{V}_i) \cap A = St(x,\underset{\sim}{U}_i) \subset V_s \cap A = U_s$. Hence $\{\underset{\sim}{U}_i\}_{i=1}^{\infty}$ is a desired sequence of open covers of A which implies A is a σ-paracompact space. Since A is an arbitrary subset of X; X is hereditarily σ-paracompact.

<u>Definition 3.1.4</u> A subset A of a topological space X is called a <u>generalized F_σ set</u> iff for each open set $U \supset A$ there is a F_σ set F such that $A \subset F \subset U$.

<u>Proposition 3.1.6</u> If X is σ-paracompact, then every generalized F_σ set A is σ-paracompact.

<u>Proof</u>. Let A be a generalized F_σ subset of X. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of A by sets open in the subspace A. Let $\underset{\sim}{V} = \{V_s \mid s \in S\}$ be a family of open sets in X such that $V_s \cap A = U_s$ for each $s \in S$. Let $V^{\#} = \cup(V_s \mid s \in S)$. Obviously, $V^{\#}$ is open in X and contains A. Since A is a generalized F_σ-set in X there exists a F_σ-set F in X such that $A \subset F \subset V^{\#}$, where

$F = \bigcup_{i=1}^{\infty} F_i$, and F_i is closed in X for each i . For each i it is obvious that $\{V_s \mid s \in S\} \cup \{(X - F_i)\}$ is an open cover of X . But, X is σ-paracompact implies for each fixed i there exists a countable family $\{\underset{\sim}{W}_j^i\}_{j=1}^{\infty}$ of open covers of X satisfying the condition that for any $x \in X$ there is j such that $St(x,\underset{\sim}{W}_j^i) \subset X - F_i$ or V_s for some $s \in S$. For arbitrary i and j define $\underset{\sim}{R}_j^i = \{W \cap A \mid w \in \underset{\sim}{W}_j^i\}$. We shall show that the countable family $\{\underset{\sim}{R}_j^i\}_{j,i=1}^{\infty}$ of open covers of A has the required property.

Let $x \in A$. Then for some i_o , $x \in F_{i_o}$ so that $x \notin X - F_{i_o}$. Now consider the countable family $\{\underset{\sim}{W}_j^{i_o}\}_{j=1}^{\infty}$; then there is j_o such that $St(x,R_{j_o}^{i_o}) = St(x,\underset{\sim}{W}_{j_o}^{i_o}) \cap A \subset V_s \cap A = U_s$ for some $s \in S$. Hence the proposition is proved.

<u>Corollary 3.1.6A</u> If X is σ-paracompact, then every F_σ subset of X is σ-paracompact.

<u>Corollary 3.1.6B</u> If X is σ-paracompact, then every closed subset of X is σ-paracompact.

<u>Corollary 3.1.6C</u> A perfectly normal σ-paracompact space is hereditarily σ-paracompact.

<u>Proposition 3.1.7</u> The sum $\bigoplus_{s \in S} X_s$ of a family $\{X_s \mid s \in S\}$ of disjoint topological spaces is σ-paracompact iff X_s is σ-paracompact for each $s \in S$.

Proof. If X is σ-paracompact, then by Corollary 3.1.6B, X_s is σ-paracompact for $s \in S$.

Conversely, let us assume that X_s is σ-paracompact for all $s \in S$ and let $\underset{\sim}{V} = \{V_t \mid t \in T\}$ be an open cover of X. For each $s \in S$ the family $\underset{\sim}{V}_s = \{X_s \cap V_t \mid t \in T\}$ is an open cover of X_s. Let $\{\underset{\sim}{V}_{s,i}\}_{i=1}^{\infty}$ be a countable family of open covers of X_s satisfying the condition that for $x \in X_s$ there is i_x such that $St(x,\underset{\sim}{V}_{s,i_x}) \subset X_s \cap V_t$ for some $t \in T$. We shall show that the countable family $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ is the required one, where $\underset{\sim}{W}_i = \cup \{\underset{\sim}{V}_{s,i} \mid s \in S\}$. For if $x \in X$, then there is s_x such that $x \in X_{s_x}$ and there is i_x such that $St(x,\underset{\sim}{V}_{s_x,i_x}) \subset X_{s_x} \cap V_t$ for some t. But $St(x,\underset{\sim}{V}_{s_x,i_x}) = St(x,\underset{\sim}{W}_{i_x})$ implies $St(x,\underset{\sim}{W}_{i_x}) \subset V_t$. Hence the proposition is proved.

A discrete space of arbitrary power is always fully normal, and hence by Proposition 3.1.1, X is σ-paracompact. We also know that every topological space is a continuous image of a discrete space, so we can easily conclude that the continuous image of a σ-paracompact space is not necessarily σ-paracompact.

Definition 3.1.5 A continuous mapping $f : X \to Y$ is called a W-mapping for any open cover W of X iff for each point $y \in Y$ there is a neighborhood 0_y of y such that $f^{-1}0_y$ is contained in some member of W.

Proposition 3.1.8 If for every open covering W of the space X

there exists an W-mapping $f : X \to Y_W$, where Y_W is σ-paracompact, then X is σ-paracompact.

Proof. Let W be an open cover of X. Then there is a continuous function $f : X \to Y_W$ where Y_W is a σ-paracompact space depending on W. Since f is an W-mapping for each $y \in Y_W$, there is an open neighborhood O_y of y such that $f^{-1}O_y$ is contained in some member of W. Obviously $\underset{\sim}{O} = \{O_y \mid y \in Y\}$ is an open cover of Y_W. Now, Y_W is σ-paracompact, so there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of Y satisfying the condition that for each $y \in Y$ there is i_y such that $St(y,\underset{\sim}{V}_{i_y})$ is contained in some member of $\underset{\sim}{O}$. We shall show that the countable family $\{f^{-1}\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X, where $f^{-1}V_i = \{f^{-1}v \mid v \in V_i\}$ for each i, has the required property.

Let $x \in X$, and $i_x = i_{f(x)}$ where $i_{f(x)}$ is an integer such that $St(f(x),\underset{\sim}{V}_{i_{f(x)}})$ is contained in some member of $\underset{\sim}{O}$. Clearly, $St(x,f^{-1}\underset{\sim}{V}_{i_x}) \subset St(f^{-1}y,f^{-1}\underset{\sim}{V}_{i_{f(x)}})$, where $y = f(x)$. It is obvious that $St(f^{-1}y,f^{-1}\underset{\sim}{V}_{i_{f(x)}})$ is contained in the inverse image of a member of $\underset{\sim}{O}$ which contains $St(y,\underset{\sim}{V}_{i_{f(x)}})$ and which in turn is contained in a member of W. Hence the proposition is proved.

Definition 3.2.1 A topological space X is F_σ-screenable iff every open cover of X has σ-discrete closed refinement.

Theorem 3.2.1 A topological space X is F_σ-screenable iff for every

open cover $\underset{\sim}{U}$ of X there is a countable family $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ of open covers of X refining $\underset{\sim}{U}$ for all i satisfying the condition that for each $x \in X$ there is i_x such that x is in exactly one member of $\underset{\sim}{W}_{i_x}$.

<u>Proof</u>. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of X . Then, there is a σ-discrete closed refinement $V = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$ of $\underset{\sim}{U}$, where for each i, $\underset{\sim}{V}_i = \{V^i \mid \alpha \in \Lambda_i\}$. Let us define $O_\alpha^i = X - \cup(V_\beta^i \mid \beta \in \Lambda_i$ and $\alpha \neq \beta)$ for $\alpha \in \Lambda_i$ and $i = 1,2,\ldots$. Obviously, O_α^i is open for each $\alpha \in \Lambda_i$, $V_\alpha^i \subset O_\alpha^i$ and $O_\alpha^i \cap V_\beta^i = \phi$, if $\alpha \neq \beta$. For each α , and i choose a set $U_s \in \underset{\sim}{U}$ such that $V_\alpha^i \subset U_s$, and denote it by H_α^i . Now for each i , let us define $\underset{\sim}{W}_i = \{O_\alpha^i \cap H_\alpha^i \mid \alpha \in \Lambda_i\} \cup \{(X - \cup(V_\alpha^i \mid \alpha \in \Lambda_i)) \cap U_s \mid s \in S\}$. It is easy to verify that for each i , $\underset{\sim}{W}_i$ is an open cover of X and if $x \in V_\alpha^i$, then x is in a unique member of $\underset{\sim}{W}_i$, namely $O_\alpha^i \cap H_\alpha^i$. We shall show that the countable family $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ of open covers of X satisfies the required property.

Let $x \in X$. Then $x \in V_\alpha^{i_x}$ for some i_x and $\alpha \in \Lambda_{i_x}$. Then by the remark in the preceding paragraph x is in a unique member of $\underset{\sim}{W}_{i_x}$. Hence $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ is the required countable family of open covers of X .

Conversely, let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of X and $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ be a countable family of open covers of X having the required property. Let $\underset{\sim}{W}_i = \{W_\alpha^i \mid \alpha \in \Lambda_i\}$ for each i . Let us define $U(\alpha,i) = \{x \in X \mid St(x,\underset{\sim}{W}_i) = W_\alpha^i\}$ for $\alpha \in \Lambda_i$ and $i = 1,2,\ldots$. The

subset $U(\alpha,i)$ of W^i_α is closed in X. For if y is a limit point of $U(\alpha,i)$ every member of $\underset{\sim}{W}_i$ which contains y will also contain a point of $U(\alpha,i)$. So, if y belongs to W^i_β for $\beta \neq \alpha$, then some point of $U(\alpha,i)$ is in two members of $\underset{\sim}{W}_i$ which is not possible, i.e, y is in $U(\alpha,i)$. Now we show that the collection $\underset{\sim}{U}_i = \{U(\alpha,i) \mid \alpha \in \Lambda_i\}$ for each i, is discrete. For if $x \in X$, there is no member of $\underset{\sim}{W}_i$ which contains x and intersects two members of the collection $\underset{\sim}{U}_i$. Hence $\underset{\sim}{U}_i$ is a discrete collection of closed sets for each i. Finally, it is easy to verify that $\bigcup_{i=1}^{\infty} \underset{\sim}{U}_i$ is a σ-discrete closed refinement of $\underset{\sim}{U}$. Hence the theorem is proved.

Corollary 3.2.1A Every F_σ-screenable space is σ-paracompact.

Corollary 3.2.1A answers a question of Arhangel'skii [4]. Recently Coban [17] and Burke and Stoltenberg [12] have also proved this Corollary.

Corollary 3.2.1B Every F_σ-screenable space is countably metacompact.

Proof. Follows from Corollary 3.2.1A and Proposition 3.1.4.

Remark 3.2.1 Every countable T_1-space is F_σ-screenable and hence σ-paracompact.

Proposition 3.2.2 Let X be an F_σ-screenable space and let F be an F_σ set of X. Then, the subspace F is F_σ-screenable.

Proof. Let $F = \bigcup_{i=1}^{\infty} F_i$, where F_i is closed in X for each i . Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of F and let us denote by $\underset{\sim}{V} = \{V_s \mid s \in S\}$ a family of open subsets of the space X such that $U_s = V_s \cap F$ for each $s \in S$. The family $\{V_s \mid s \in S\} \cup \{(X - F_i)\}$ is an open cover of X for each i . X is F_σ-screenable, so for each i there exists a σ-discrete closed refinement $\underset{\sim}{W}^i = \bigcup_{j=1}^{\infty} \underset{\sim}{W}_j^i$. Let us define $\underset{\sim}{V}_j^i = \{W \cap F_i \mid w \in \underset{\sim}{W}_j^i$ and $W \cap F_i \neq \phi\}$. It is easy to verify that $\bigcup_{i,j=1}^{\infty} \underset{\sim}{V}_j^i$ is a σ-discrete closed refinement of $\underset{\sim}{U}$. Hence F is F_σ-screenable.

Corollary 3.2.2A Every closed subspace of a F_σ-screenable space is F_σ-screenable.

Proposition 3.2.3 The sum $\bigoplus_{s \in S} X_s$ of the family $\{X_s \mid s \in S\}$ of disjoint topological spaces is F_σ-screenable iff each X_s is F_σ-screenable.

Proof. Similar to Proposition 3.1.7.

Proposition 3.2.4 Let X be a topological space such that $X = \bigcup_{i=1}^{\infty} X_i$, where X_i for each i is a closed F_σ-screenable subspace of X . Then X is F_σ-screenable.

Proof. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of X . Then, for each i , the family $\{U_s \cap X_i \mid s \in S\}$ is an open cover of X_i and so there exists a σ-discrete closed refinement $\underset{\sim}{V}_i = \bigcup_{n=1}^{\infty} \underset{\sim}{V}_{i,n}$ in X_i . Since

X_i is closed, for each i and arbitrary n, $\underset{\sim}{V}_{i,n}$ is a discrete collection of closed subsets of X. Now, it is clear that $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i = \bigcup_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \underset{\sim}{V}_{i,n}$ is a required σ-discrete closed refinement of $\underset{\sim}{U}$. Hence X is F_σ-screenable.

Corollary 3.2.4A Let X be a regular space which is a countable union of closed paracompact spaces. Then X is F_σ-screenable.

Proof. Follows by Proposition 3.2.4 and the fact that a regular paracompact space is F_σ-screenable.

Proposition 3.2.5 If X is a Hausdorff locally compact and F_σ-screenable space, then X is a countable union of closed paracompact subspaces.

Proof. Let $\underset{\sim}{U}$ be an open cover of X and $\underset{\sim}{V}$ be an open refinement such that the closure of each of its members is compact. X is F_σ-screenable, so there exists a σ-discrete closed refinement $\underset{\sim}{W} = \bigcup_{i=1}^{\infty} \underset{\sim}{W}_i$ of $\underset{\sim}{V}$, where $\underset{\sim}{W}_i = \{W^i \mid \alpha \in \Lambda\}$ for each i. We shall show that for each i the subset $\underset{\sim}{W}_i^{\#} = \cup (W_\alpha^i \mid \alpha \in \Lambda_i)$ is paracompact as a subspace of X. By Theorem 2 of Suzuki [54] $W_i^{\#}$ is collectionwise normal and by Corollary 3.2.4A, $W_i^{\#}$ is F_σ-screenable. Finally be Lemma 2 of McAuley [34] $W_i^{\#}$ is paracompact for each i. Hence the proposition is proved.

Remark 3.2.2 From Proposition 3.2.4 and Proposition 3.2.5 we immediately conclude that a locally compact Hausdorff space is

F_σ-screenable iff it can be represented as a countable union of closed paracompact subspaces.

Theorem 3.2.6[†] A topological space X is F_σ-screenable iff every open cover of X has a σ-locally finite closed refinement.

Proof. Clearly, if X is F_σ-screenable then every open cover of X has a σ-locally finite closed refinement.

Conversely, suppose every open cover of X has a σ-locally finite closed refinement. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of X and let $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$, where $\underset{\sim}{V}_i = \{V^i \mid \alpha \in \Lambda_i\}$ for each i, be a σ-locally finite closed refinement of $\underset{\sim}{U}$. For each i and $\alpha \in \Lambda_i$ let us denote by s^i_α a fixed index in S such that $V^i_\alpha \subset U_{s^i_\alpha}$. Now we denote by $W(V^i_\alpha) = (U_{s^i_\alpha} \times U_{s^i_\alpha}) \cup ((X - V^i_\alpha) \times (X - V^i_\alpha))$ and $W_i = \cap(W(V^i) \mid \alpha \in \Lambda_i)$ for each i. We claim that for each i, W_i is a symmetric neighborhood of $\Delta = \{(x,x) \mid x \in X\}$. Let $(x,x) \in \Delta$ and 0^i_x be a neighborhood of x in X which intersects at most finitely many members of $\underset{\sim}{V}_i$ say $V^i_{\alpha_1}, \ldots, V^i_{\alpha_n}$. This is possible as $\underset{\sim}{V}_i$ is locally finite. Clearly $0^i_x \times 0^i_x \subset (X - V^i_\beta) \times (X - V^i_\beta)$ for all $\beta \in \Lambda_i$ for which $0^i_x \cap V^i_\beta = \phi$. Now, it is easy to see that $(0^i_x \times 0^i_x) \cap (\bigcap_{j=1}^{n} W(V^i_{\alpha_j}))$ is a neighborhood of (x,x) and is contained in W_i.

Without loss of generality let us assume that $\underset{\sim}{V}_i \subset \underset{\sim}{V}_{i+1}$ for each i. Then by the construction, it is easy to see that for each i, W_i is symmetric, $W_{i+1} \subset W_i$ and $\Delta \subset W_i$. On X let

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[†] D.K. Burke, On subparacompact spaces, Proc. Amer. Math. Soc., 23 (1969), 655-663.

us define a new topology τ such that $U \in \tau$ iff for each $x \in U$ there i such that $W_i[x] \subset U$. It is easy to see that τ is a topology. Let us define $d : X \times X \to R$, where R denotes the set of real numbers, by setting $d(x,y) = \frac{1}{2^i}$ iff $(x,y) \in W_i - W_{i+1}$ for $i = 0,1,2,\ldots$ and $d(x,y) = 0$, otherwise. Obviously, $d(x,y) = d(y,x)$, $d(x,y) = 0$ if $x = y$ and $x \in Cl_\tau A \iff \inf\{d(x,A)\} = 0$.

Now, for $V_i^{\#} = \cup(V_\alpha^i \mid \alpha \in \Lambda_i)$, consider $\underset{\sim}{W} = \{int_\tau (W_i[x]) \mid x \in V_i^{\#}\}_{i=1}^{\infty}$. This is an open cover of X in the topology τ, for each $x \in int_\tau (W_i[x])$. Also, for each i and $x \in V_i^{\#}$ we have $W_i[x] \subset W(V_\alpha^i) \subset U_{s_\alpha^i}$ for some $\alpha \in \Lambda_i$, i.e., $\underset{\sim}{W}$ is a refinement. Now using techniques of Lemma 1 in McAuley [34] one can show that every τ open cover of X has a σ-discrete closed refinement with respect to τ. Since τ is weaker than the original topology, σ-discrete closed collection in τ is also a σ-discrete closed collection in the original topology. Hence the theorem is proved.

Definition 3.2.5 A mapping $f : X \to Y$ is called perfect iff f is continuous, closed and for each $y \in Y$, $f^{-1}y$ is compact.

Theorem 3.2.7 Let X and Y be regular spaces and $f : X \to Y$ be a perfect onto mapping. Then X is F_σ-screenable iff Y is F_σ-screenable.

<u>Proof</u>. Let X be F_σ-screenable space and f be a perfect mapping of X onto Y. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of Y. By the continuity of f the collection $\{f^{-1}U_s \mid s \in S\}$ is an open cover of X. Let $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$ be a σ-discrete closed refinement of $\{f^{-1}U_s \mid s \in S\}$, where $\underset{\sim}{V}_i = \{V^i \mid \alpha \in \Lambda_i\}$ for each i. Let us now define $\underset{\sim}{W} = \bigcup_{i=1}^{\infty} \underset{\sim}{W}_i$ where $\underset{\sim}{W}_i = \{f(V^i) \mid \alpha \in \Lambda_i\}$ for $i = 1,2,\ldots$. Since for each $y \in Y$, $f^{-1}y$ is compact. For fixed i there is an open set $O_i \supset f^{-1}y$ and intersecting only finitely many members of $\underset{\sim}{V}_i$. By the fact that f is continuous and closed by Theorem 11.2, page 86 of Dugundji [18] there exists a neighborhood G_i of y such that $f^{-1}G_i \subset O_i$. Clearly, if O_i intersects finitely many members of $\underset{\sim}{V}_i$, so does G_i intersect finitely many members of $\underset{\sim}{W}_i$. Hence $\underset{\sim}{W} = \bigcup_{i=1}^{\infty} \underset{\sim}{W}_i$ is a σ-locally finite closed refinement of $\underset{\sim}{U}$. Consequently, by Theorem 3.2.6, Y is F_σ-screenable.

Conversely, let f be a perfect mapping of X onto a F_σ-screenable space Y. Let $\underset{\sim}{W}$ be any open cover of X and $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open refinement of $\underset{\sim}{W}$ such that the cover by closures of members of $\underset{\sim}{U}$ also refines $\underset{\sim}{W}$. This is possible as X is regular. For each $y \in Y$, $f^{-1}y$ is compact so we may pick a finite set $s(y) \subset S$ say $S_1^y,\ldots,S_{n(y)}^y$ such that $f^{-1}y \subset \bigcup_{i=1}^{n(y)} U_{s_i^y}$. Let $V_y = Y - f(X - \bigcup_{i=1}^{n(y)} U_{s_i^y})$. Then, since f is closed V_y is an open neighborhood of y such that $f^{-1}V_y \subset \bigcup_{i=1}^{n(y)} U_{s_i^y}$. Now let $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$, where for each i, $\underset{\sim}{V}_i = \{V_\alpha^i \mid \alpha \in \Lambda_i\}$, be a σ-discrete closed reinement of $\{V_s \mid s \in S\}$. For each i and $\alpha \in \Lambda_i$ pick

y such that $V^i \subset V_y$ and let $s_j^y \in S(y)$. Then, define $W(i,\alpha,s_j^y) = f^{-1}V_\alpha^i \cap U_{s_j^y}$. For fixed i and j the collection $\{W(i,\alpha,s_j^y) \mid \alpha \in \wedge_i\}$ is discrete as for fixed i , $\underset{\sim}{V}_i$ is discrete and f is continuous. Let us define $\underset{\sim}{W}^1 = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{W(i,\alpha,s_j^y) \mid \alpha \in \wedge_i\}$. It is easy to see that $\underset{\sim}{W}^1$ is a refinement of $\underset{\sim}{U}$ and is σ-discrete. Consequently, closures of members of $\underset{\sim}{W}^1$ from a σ-discrete closed refinement of $\underset{\sim}{W}$. Hence the theorem is proved.

<u>Definition 3.3.1</u> A topological space X is <u>σ-fully normal</u> iff every open cover of X has an open σ-point star refinement.

<u>Definition 3.3.2</u> A topological space X is <u>screenable</u> iff every open cover of X has a open σ-disjoint refinement.

<u>Definition 3.3.3</u> A topological space X is <u>σ-metacompact</u> iff every open cover of X has an open σ-point finite refinement.

<u>Definition 3.3.4</u> A topological space X is <u>meta-Lindelöf</u> iff every open cover of X has an open point countable refinement.

<u>Proposition 3.3.1</u> A topological space X is metacompact iff it is σ-metacompact and countably metacompact.

<u>Proof</u>. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of X and $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \{V_\alpha^i \mid \alpha \in \wedge_i\}$ be an open σ-point finite refinement of $\underset{\sim}{U}$.

Let $V_i^{\#} = \cup(V^i \mid \alpha \in \wedge_i)$ for each i . Then $\{V_i^{\#}\}_{i=1}^{\infty}$ is a countable open cover of X . But X is countably metacompact, so that by Theorem 1.4 on page 142 of Dugundji [17], there exists an open point finite refinement $\underset{\sim}{W} = \{W_i\}_{i=1}^{\infty}$ such that $W_i \subset V_i^{\#}$ for each i . Let us define $\underset{\sim}{B}_i = \{V_\alpha^i \cap W_i \mid \alpha \in \wedge_i\}$ for $i = 1,2,...,$ and $\underset{\sim}{B} = \bigcup_{i=1}^{\infty} \underset{\sim}{B}_i$. It is easy to verify that $\underset{\sim}{B}$ is an open refinement of $\underset{\sim}{U}$. Now we show that $\underset{\sim}{B}$ is point finite.

Let $x \in X$. Then there are only finitely many indices $i_1,...,i_{n(x)}$ such that $x \in W_{i_j}$ for $j = 1,...,n(x)$ only. But each $\underset{\sim}{B}_{i_j}$ is point finite, so x belongs to only finitely many members of $\underset{\sim}{B}$. Therefore $\underset{\sim}{B}$ is an open point finite refinement of $\underset{\sim}{U}$. Hence X is metacompact.

The converse is obvious.

Corollary 3.3.1A A σ-paracompact (or F_σ-screenable) and σ-metacompact space is metacompact.

Proof. Follows from Proposition 3.1.4, Corollary 3.2.1B and Proposition 3.3.1.

Corollary 3.3.1B A topological space is countably metacompact iff every σ-point finite open cover has an open point finite refinement.

Remark 3.3.1 Proposition 3.3.1 can also be found in Kljushin [30] and Greever [21].

Proposition 3.3.2 Let X be metacompact and σ-paracompact. Then every open cover of X has a σ-locally finite closed refinement.

Proof. It is enough to show that every point finite open cover $\underset{\sim}{U} = \{U_s \mid s \in S\}$ of X has a σ-locally finite closed refinement. Since X is metacompact and σ-paracompact there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X with the property that for each i, $\underset{\sim}{V}_i$ refines $\underset{\sim}{U}$ and is point finite. Furthermore, for each $x \in X$ there is i_x such that $St(x, \underset{\sim}{V}_{i_x}) \subset U$ for some $U \in \underset{\sim}{U}$. Let us define for each $i = 1,2,\ldots$ and $s \in S$, $W(s,i) = \{x \in X \mid St(x,\underset{\sim}{V}_i) \subset U_s\}$. $W(s,i)$ is closed for each i and s. For, if y is a limit point of $W(s,i)$ the open neighborhood $O_y = \cap\, (V \in \underset{\sim}{V}_i \mid y \in V)$ of y has a non-empty intersection with $W(s,i)$ which inplies $y \in W(s,i)$ and therefore $W(s,i)$ is closed in X. Let us define $\underset{\sim}{W}_i = \{W(s,i) \mid s \in S\}$ for each i. $\underset{\sim}{W}_i$ is locally finite, because if $x \in X$, $O_x = \cap\, (V \in \underset{\sim}{V}_i \;\; x \in V)$ is an open neighborhood of x which does not intersect more then finitely many members of $\underset{\sim}{W}_i$. For if O_x intersects infinitely many members of $\underset{\sim}{W}_i$ then x will be contained in infinitely many members of $\underset{\sim}{U}$ which contradicts the fact that $\underset{\sim}{U}$ is point finite. It is now easy to see that $\underset{\sim}{W} = \bigcup_{i=1}^{\infty} \underset{\sim}{W}_i$ is a required σ-locally finite closed refinement of $\underset{\sim}{U}$.

Corollary 3.3.2A Every metacompact and σ-paracompact space is F_σ-screenable.

Proof. Follows from Proposition 3.3.2 and Theorem 3.2.6.

Corollary 3.3.2B Every compact (or Lindelöf) σ-paracompact space is F_σ-screenable.

Theorem 3.3.3[†] If X is a regular metacompact space in which every closed set is a G_δ set, then X is F_σ-screenable.

Proof. Let $\underset{\sim}{U}$ be an open cover of X and $\underset{\sim}{V}^1$ be an open point finite refinement of $\underset{\sim}{U}$. Let M_1 be the set of all points of X each of which belongs to exactly one member of $\underset{\sim}{V}^1$. For each $V \in \underset{\sim}{V}^1$ such that $V \cap M_1 \neq \phi$ let us define $R_1(V) = V \cap M_1$. It is easy to verify that $\{R_1(V) \mid V \in \underset{\sim}{V}^1$ and $V \cap M_1 \neq \phi\}$ is a discrete collection and $M_1 = \cup\, (R_1(V) \mid V \in \underset{\sim}{V}^1$ and $V \cap M_1 \neq \phi)$. M_1 is closed so $M_1 = \bigcap_{i=1}^{\infty} G_i^1$, where G_i^1 is open in X for each i. Let M_2 be the set of all points of X each of which belongs to exactly two members of $\underset{\sim}{V}^1$ and let $K_2^i = \{g : g = h \cap k$ for $h \neq k$ and $h,k \in \underset{\sim}{V}^1\} \cup \{G_i^1\}$ for each i. If $p \in X - M_1$ then p belongs to at least two members of $\underset{\sim}{V}^1$. Hence $\underset{\sim}{K}_2^1$ covers X for each i. Now, if $p \in M_2$ then p belongs to exactly one member of $K_2^{i_p}$ for some i_p as $\bigcap_{i=1}^{\infty} G_i^1 = M_1$. Let us denote by M_{2i} the set of all $x \in M_2$ which are exactly in one member of K_2^i and hence not in G_i^1. It is clear that $M_2 = \bigcup_{i=1}^{\infty} M_{2i}$. Now as before let us construct $R_{2i}(V) = V \cap M_{2i}$ for $V \in K_2^i$ and for each i. We can easily verify that for each i the collection $\{R_{2i}(V) \mid V \in K_2^i,\ V \cap M_{2i} \neq \phi\}$ is discrete, and $\cup(R_{2i}(V) \mid V \in K_2^i,\ V \cap M_{2i} \neq \phi) = M_{2i}$ for each i.

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† R.E. Hodel, A note on subparacompact spaces, Proc. Amer. Math. Soc., 25(1970), 842-845.

For $n > 2$, let M_n be the set of all points belonging to exactly n members of $\underset{\sim}{V}^1$, let $D_{n-1} = \bigcup_{i=1}^{n-1} M_i$. D_{n-1} is closed so $D_{n-1} = \bigcap_{i=1}^{\infty} G_i^{n-1}$ where G_i is open in X for each i. Let us define $K_n^i = \{g \mid g = \bigcap_{i=1}^{n} h_i , h_i \neq h_j$ for $i \neq j$, $h_i \in \underset{\sim}{V}^1$ for $i = 1,\ldots,n\} \cup \{G_i^{n-1}\}$. Let us denote by M_{ni} the set of $x \in M_n$ which are exactly in one member of K_n^i. It is clear that $M_n = \bigcup_{n=1}^{\infty} M_{ni}$ and as before for each i, we define

$$R_{ni}(V) = V \cap M_{ni} \quad \text{for } V \in K_n^i \text{ and for } V \cap M_{ni}^i \neq \phi \;.$$

We can easily see that $X = M_1 \cup (\bigcup_{j=2}^{\infty} \bigcup_{i=1}^{\infty} M_{ji})$ and

$\underset{\sim}{W} = \{R_1(V) \mid V \in V^1, V \cap M \neq \phi\} \cup \bigcup_{j=2}^{\infty} \bigcup_{j=1}^{\infty} \{R_{ji}(V) \mid V \in K_j^i, V \cap M_{ji} \neq \phi\}$

is a σ-discrete refinement of $\underset{\sim}{U}$. But X is regular, so we can conclude that every open cover of X has a σ-discrete closed refinement. Hence the theorem is proved.

<u>Theorem 3.3.4</u> A countably metacompact space X is σ-fully normal iff it is screenable.

<u>Proof</u>. A screenable space is obviously a σ-fully normal space.

Conversely, let X be a countably metacompact and σ-fully normal space. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of X and $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$, where $\underset{\sim}{V}_i = \{V_\alpha^i \mid \alpha \in \wedge_i\}$ be an open σ-point star refinement of $\underset{\sim}{U}$. Let us denote by $V_i^{\#} = \cup\, (V_\alpha^i \mid \alpha \in \wedge_i)$ for each i. Clearly $\{V_i^{\#}\}_{i=1}^{\infty}$ is a countable open cover of X. Since X is countably metacompact, there exists an open point finite refinement

$\{W_i\}_{i=1}^{\infty}$ such that $W_i \subset V_i^{\#}$ for each i. Now, let us define $\underset{\sim}{R}_i = \{W_i \cap V_\alpha^i \mid \alpha \in \Lambda_i\}$. Obviously, $\underset{\sim}{R} = \bigcup_{i=1}^{\infty} \underset{\sim}{R}_i$ is an open refinement of $\underset{\sim}{U}$ and for each $x \in X$ there exists a finite collection $U_{s_1}, \ldots, U_{s_n}$ of members of $\underset{\sim}{U}$ such that every member of $\underset{\sim}{R}$ which contains x belongs to one of them. Now, by Theorem 1 of Worrell [58] X is metacompact and finally by Theorem 6 of Heath [24], X is screenable.

<u>Corollary 3.3.4A</u> A σ-paracompact (or F_σ-screenable) space is screenable iff X is σ-fully normal.

<u>Proof</u>. Follows from Proposition 3.1.4 and Theorem 3.3.4.

<u>Definition 3.3.5</u> A topological space X is called <u>weakly collectionwise normal</u> iff for every discrete family $\{C_\alpha \mid \alpha \in \Lambda\}$ of closed sets there exists a point finite collection $\{O_\alpha \mid \alpha \in \Lambda\}$ of open sets such that for each $\alpha \in \Lambda$, $C_\alpha \subset O_\alpha$ and $O_\alpha \cap C_\beta = \phi$, if $\alpha \neq \beta$.

<u>Proposition 3.3.5</u> Every weakly collectionwise normal F_σ-screenable space is metacompact.

<u>Proof</u>. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of X and $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$, where $\underset{\sim}{V}_i = \{V_\alpha^i \mid \alpha \in \Lambda_i\}$ be a σ-discrete closed refinement of $\underset{\sim}{U}$. Since X is weakly collectionwise normal, for each i there exists an open point finite collection $\{O_\alpha^i \mid \alpha \in \Lambda_i\}$ such that $O_\alpha^i \supset V_\alpha^i$ for each $\alpha \in \Lambda_i$ and $V_\beta^i \cap O_\alpha^i = \phi$ if $\alpha \neq \beta$.

Let us denote by $H_\alpha^i = O_\alpha^i \cap U_s$ where $U_s \in \underset{\sim}{U}$ and is such that $V_\alpha^i \subset U_s$. Now, it is easy to verify that $\bigcup_{i=1}^{\infty} \{H_\alpha^i \mid \alpha \in \Lambda_i\}$ is an open σ-point finite refinement of $\underset{\sim}{U}$, i.e., X is σ-metacompact. Hence by Corollary 3.3.1A, X is metacompact.

<u>Remark 3.3.2</u> Every normal metacompact or collectionwise normal space is weakly collectionwise normal. On the otherhand there are collectionwise normal spaces which are not metacompact. Therefore, weakly collectionwise normal spaces need not be metacompact.

<u>Proposition 3.3.6</u> A screenable space X is F_σ-screenable iff X is σ-paracompact.

<u>Proof</u>. The necessity follows from Corollary 3.2.1A. For sufficiency, observe that every screenable space is σ-metacompact. Therefore by Proposition 3.1.4 and Proposition 3.3.1 a screenable and σ-paracompact space is metacompact. Now, finally by Proposition 3.3.2 and Theorem 3.2.6, X is F_σ-screenable.

<u>Proposition 3.3.7</u> Let X be a σ-fully normal space in which every closed set is a G_δ set. Then X is σ-paracompact.

<u>Proof</u>. Let $\underset{\sim}{U}$ be an open cover of X and $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$ be an open σ-point star refinement of $\underset{\sim}{U}$. Let us denote by $V_i^{\#} = \cup\,(V \mid V \in \underset{\sim}{V}_i)$ for each i . Since every closed set is a G_δ , set $X - V_i^{\#} = \bigcap_{j=1}^{\infty} G_j^i$, where G_i^i is open in X . Let us define

$\underset{\sim}{W}_j^i = \{V \mid V \in \underset{\sim}{V}_i\} \cup \{G_j^i\}$ for each i and j. Now, it is easy to show that the countable family $\{\underset{\sim}{W}_j^i\}_{i,j=1}^{\infty}$ of open covers of X has the required property.

Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be collections of subsets of a set X, and m be any cardinal number $\geqq 2$. $\underset{\sim}{B}$ is an <u>m-star refinement</u> of $\underset{\sim}{A}$ if (i) for each $\underset{\sim}{C} \subset \underset{\sim}{B}$ with $|\underset{\sim}{C}| \leqq m$ and $\cap (C \mid C \in \underset{\sim}{C}) \neq \phi$, there is $A \in \underset{\sim}{A}$ such that $\cup (C \mid C \in \underset{\sim}{C}) \subset A$.

Let m be any cardinal number $\geqq 2$.

<u>Definition 3.3.6</u> (Mansfield [33]) A topological space X is <u>m-fully normal</u> iff each open cover of X has an open m-star refinement.

<u>Theorem 3.3.8</u> (Arhangelskii [6]) In a topolgoical space X, the following statements are equivalent:

(i) X is fully normal;

(ii) every open cover of X has an open star refinement;

(iii) for every open cover $\underset{\sim}{U}$ of X there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the condition that for any $x \in X$ there is n_x and a neighborhood O_x of x such that $St(O_x, \underset{\sim}{V}_{n_x}) \subset U$ for some $U \in \underset{\sim}{U}$.

<u>Proof</u>. (i) ⟹ (ii) see Theorem 3.4 on page 157 of Dugundji [18].

(ii) ⟹ (iii) is obvious.

(iii) ⟹ (i) see Theorem 3.7 on page 169 of Dugundji [18].

Thoerem 3.3.9 (A.H. Stone [53]) A topological space X is Hausdorff and paracompact iff X is a fully normal and T_1-space.

Proof. See Theorem 3.5 on page 169 of Dugundji [18].

Theorem 3.3.10 A topological space X is fully normal iff X is 2-fully normal and σ-paracomapct.

Proof. First, we show that 2-fully normal and σ-paracompact space is fully normal. Let $\underset{\sim}{U}$ be an open cover of X. Then, by the σ-paracompactness of X there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the condition that for any $x \in X$ there is n_x such that $St(x,\underset{\sim}{V}_{n_x}) \subset \underset{\sim}{U}$ for some $U \in \underset{\sim}{U}$. Since X is also 2-fully normal for each i, $\underset{\sim}{V}_i$ has an open 2-star refinement say $\underset{\sim}{W}_i$. We shall show that the countable family $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ of open covers of X satisfies the condition that for each $x \in X$ there is i_x and a neighborhood O_x of x such that $St(O_x,\underset{\sim}{V}_{i_x}) \subset U$ for some $U \in \underset{\sim}{U}$.

Let x_o be any point of X and n_{x_o} be such that $St(x_o,\underset{\sim}{V}_{n_{x_o}}) \subset U$ for some $U \in \underset{\sim}{U}$. Let us pick $W_o \in \underset{\sim}{W}_{n_{x_o}}$ such that $x_o \in W_o$. Now, for $W \in \underset{\sim}{W}_{n_{x_o}}$ such that $W \cap W_o \neq \phi$, $W \cup W_o \subset V$ for $V \in \underset{\sim}{V}_{n_{x_o}}$, but each such $V \in \underset{\sim}{V}_{n_{x_o}}$ contains x_o. Consequently, we have $St(W_o,\underset{\sim}{W}_{n_{x_o}}) \subset St(x_o,\underset{\sim}{V}_{n_{x_o}}) \subseteq U$ for some $U \in \underset{\sim}{U}$.

Now, by taking $O_{x_o} = W_o$ and $i_{x_o} = n_{x_o}$ we have proved the claim. Hence by Theorem 3.3.8, X is a fully normal space.

The converse is obvious.

<u>Corollary 3.3.10A</u> A topological space X is Hausdorff and paracompact iff it is a T_1, 2- fully normal and σ-paracompact space.

<u>Remark 3.3.3</u> In Hausdorff 2-fully normal spaces, it is not difficult to show that σ-paracompact, metacompact and meta-Lindelöf properties are equivalent. This remark follows from Theorem 3.3.10 and Theorem 5.10 of Mausfield [33].

<u>Proposition 3.3.11</u> Let X be a T_1, σ-paracompact space in which every uncountable set has a limit point. Then X is Lindelöf.

<u>Proof</u>. Let $\underset{\sim}{U}$ be an open cover of X. Then by Proposition 3.1.3 there exists a σ-cushioned closed refinement $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$ of $\underset{\sim}{U}$. If every uncountable set has a limit point, then $\underset{\sim}{V}_i$ is at most a countable collection for each i. Now, it follows immediately that $\underset{\sim}{U}$ has countable subcover. Hence X is Lindelöf.

<u>Corollary 3.3.11A</u> Every T_1, σ-paracompact and countably compact space is compact.

<u>Proposition 3.3.12</u> Let X be a T_1, σ-paracompact and hereditarily separable space. Then X is Lindelöf.

Proof. If X is hereditarily separable, then every uncountable set has a limit point. Now by Proposition 3.3.11, X is Lindelöf.

Proposition 3.3.13 Let X be a normal, screenable and σ-paracompact space. Then X is paracompact.

Proof. Let $\underset{\sim}{U} = \{U_s \mid s \in S\}$ be an open cover of X and $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$ where $\underset{\sim}{V}_i = \{V_\alpha^i \mid \alpha \in \Lambda_i\}$ be a σ-disjoint open refinement of $\underset{\sim}{U}$. Then using methods employed in Proposition 3.1.3 we can construct $\underset{\sim}{W} = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \underset{\sim}{W}_j^i$, where $\underset{\sim}{W}_j^i = \{W_\alpha^{i,j} \mid \alpha \in \Lambda_i\}$ is a σ-discrete closed refinement of $\underset{\sim}{U}$ and for each fixed i and any j we have $W_\alpha^{i,j} \subset V_\alpha^i$ for $\alpha \in \Lambda_i$. Since X is normal by Lemma 1 of Michael [38], there exists a σ-discrete open refinement of $\underset{\sim}{U}$. Finally by Kelley [28] X is paracompact.

Remark 3.3.4 In Proposition 3.3.13 normality cannot be replaced by complete regularity. See Example 2 on page 766 of Heath [24].

Proposition 3.3.14 Let X be a weakly collectionwise normal, F_σ-screenable, connected, locally connected peripherally separable space. Then X is Lindelöf.

Proof. Follows from Proposition 3.3.5 and Theorem 4 of Grace and Heath [22].

Proposition 3.3.15 A completely regular space X, is σ-paracompact

and a M-space iff it is a paracompact p-space.

Proof. If X is σ-paracompact and a M-space then by Theorem 6.1 of Morita [41], Corollary 3.3.11A and Theorem 5.2 of Arhangel'skii [4], X is a paracompact p-space.

The converse follows from Corollary 3.3.10A, Theorem 5.2 of Arhangel'skii [4] and Theorem 6.1 of Morita [41].

Proposition 3.3.16 Every σ-paracompact space with G_δ-diagonal has $\bar{G}_\delta$-diagonal.

Proof. Is obvious.

Example 3.4.1 There exists a finite T_o-space which is not a σ-paracompact space.

Example 3.4.2 An uncountable space with the cofinite topology (in which the closed sets are the finite sets) is an example of a compact T_1-space which is not a σ-paracompact space.

Example 3.4.3 There exists a regular metacompact space which is not a σ-paracompact space.

Let Ω_1 denote the first uncountable ordinal and let $X_1 = \{\alpha \mid \alpha \leqq \Omega_1\}$ with the discrete topology except at Ω_1. Let the topology at Ω_1 be the ordered topology. Let Ω_2 denote the least

ordinal such that if $X_2 = \{\beta \mid \beta \leqq \Omega_2\}$ the cardinality of X_2 is greater than the cardinality of X_1. Let X_2 have discrete topology except at Ω_2. Let the topology at Ω_2 be the ordered topology. Let $X = X_1 \times X_2 - (\Omega_1, \Omega_2)$ with the product topology.

It is easy to see that X is completely regular because there is a base for the topology of X which consists of sets which are at the same time both open and closed. We now show that X is metacompact.

Let $\underset{\sim}{U}$ be an open cover of X and let $\underset{\sim}{V} = \{V(\alpha,\beta) \mid (\alpha,\beta) \in X\}$ denote the following refinement of $\underset{\sim}{U}$:

(a) if $\alpha \neq \Omega_1$, $\beta \neq \Omega_2$, let $V(\alpha,\beta) = (\alpha,\beta)$;

(b) if $\alpha = \Omega_1$, $\beta \neq \Omega_2$, let $W(\Omega_1,\beta)$ denote an element of $\underset{\sim}{U}$ containing (Ω_1,β) and let $V(\Omega_1,\beta) = \{(\gamma,\beta) \mid 1 \leqq \gamma \leqq \Omega_1\} \cap W(\Omega_1,\beta)$;

(c) if $\alpha \neq \Omega_1$, $\beta = \Omega_2$, let $W(\alpha,\Omega_2)$ denote an element of $\underset{\sim}{U}$ containing (α,Ω_2) and let $V(\alpha,\Omega_2) = \{(\alpha,\gamma) \mid 1 \leqq \gamma \leqq \Omega_2 \cap W(\alpha,\Omega_2)$. Clearly, $\underset{\sim}{V}$ is an open cover of X refining $\underset{\sim}{U}$. Moreover, each point of X belongs to atmost three points of $\underset{\sim}{V}$. Hence X is metacompact. Now it remains to show that X is not σ-paracompact, for which inview of Corollary 3.3.2A it will be enough to show that X is not F_σ-screenable.

Let $\underset{\sim}{U} = \{U(\alpha,\beta) \mid (\alpha,\beta)$ is of the form (Ω_1,β), (α,Ω_2) or $(\Omega_1,\Omega_2)\}$, define

(i) if $\alpha = \Omega_1$, $\beta \neq \Omega_2$, let $U(\alpha,\beta) = \{(\gamma,\beta) \mid 1 \leqq \gamma \leqq \Omega_1\}$;

(ii) if $\alpha \neq \Omega_1$, $\beta = \Omega_2$, $U(\alpha,\beta) = \{(\alpha,\gamma) \mid 1 \leqq \gamma \leqq \Omega_2\}$;

and

(iii) if $\alpha = \Omega_1$, $\beta = \Omega_2$, $U(\alpha,\beta) = \{(\alpha,\beta) \in X \mid$ either $\alpha = \Omega_1$ or $\beta = \Omega_2\}$.

Suppose there exists a σ-discrete closed refinement $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$, where $\underset{\sim}{V}_i = \{V_\alpha^i \mid \alpha \in \Lambda_i\}$ for each i , of $\underset{\sim}{U}$. For each $U \in \underset{\sim}{U}$ of type (i) there exists j such that $|V_j^\# \cap U| = \aleph_1$ where $V_j^\# = \cup (V^j \mid \alpha \in \Lambda_j)$. This implies that the member (x,y) of $V_j^\#$ such that $x = \Omega_1$ is a limit point of $V_j^\# \cap U$, hence some member of $\underset{\sim}{V}_i$ contains all except countably many points of $V_j^\# \cap U$. Therefore for some j and some subcollection $\underset{\sim}{V}_j^1$ of $\underset{\sim}{V}_j$ having cardinal $\aleph_2$ it is true that if V_α^j belongs to $\underset{\sim}{V}_j^1$ then some element of $\underset{\sim}{U}$ of type (i) contains $\aleph_1$ points of V^j . Since $\aleph_2 > \aleph_1$ shows that for some $\alpha_o < \Omega_1$ it is true that $\aleph_2$ elements of $\underset{\sim}{V}_j^1$ contains (x,y) such that $x = \alpha_o$. Hence every neighborhood of (α_o,Ω_2) intersects $\aleph_2$ members of $\underset{\sim}{V}_j^1$, which is a contradiction.

<u>Example 3.4.4</u> There exist a perfectly normal F_σ- screenable spaces which are not metacompact.

Example 4 of Bing [11] is a perfectly normal space which is a countable union of closed metrizable spaces, therefore by Proposition 3.2.4, X is F_σ-screenable. But by Michael [37], X is not a metacompact space.

<u>Example 3.4.5</u> There exists a perfectly normal, F_σ-screenable and metacompact space which is not paracompact.

See Example 2 of Michael [37].

CHAPTER IV

SEMI-METRIC SPACES

Apparently, little attention was given to the study of semi-metric spaces before F.B. Jones began a systemetic study of them [27]. One should, of course, note the work done by Frechet [20], and Wilson [57]. Recently, several other mathematicians became involved in this field.

In Section 1, we list some of the simple well-known properties of semi-metric spaces and give several characterizations of semi-metrizable spaces.

In Section 2, we give various mapping theorems for semi-metric spaces.

Definition 4.1.1 A T_1-space X is a semi-metric space iff there exists a non-negative real valued function d on $X \times X$ such that for any $(x,y) \in X \times X$

(1) $d(x,y) = 0$ if $x = y$;

(2) $d(x,y) = d(y,x)$;

and

(3) for any $A \subset X$, $x \in A^-$ iff $\inf \{d(x,y) \mid y \in A\} = 0$.

For each $x \in X$ and positive nubmer $\epsilon > 0$, we shall always denote the set $\{y \mid d(x,y) < \epsilon\}$ by $S(x,\epsilon)$ and shall call

$S(x,\varepsilon)$ the sphere of radius ε about x .

Remark 4.1.1 There exists a regular semi-metric space for which there is no semi-metric under which all spheres are open. See Heath [25].

Proposition 4.1.1 Let (X,d) be a semi-metric space. Then, for each $x \in X$ and any positive number $\varepsilon > 0$; $x \in \text{int}\,(S(x,\varepsilon))$.

Proof. Let $\varepsilon > 0$ be given. If $A = X - S(x,\varepsilon)$, then $d(x,A) > 0$, so $x \notin A^-$. Then $x \in X - X^- \subset S(x,\varepsilon)$, so that $x \in \text{int}\,(S(x,\varepsilon))$.

Proposition 4.1.2 Let (X,d) be a semi-metric space. If $A_{m,n}(x) = \{z \in X \mid S(z,m) \subset S(x,n)\}$, then $A^-_{m,n}(x) \subset S(x,n)$ for fixed m and n .

Proof. Let y be a limit point of $A_{m,n}(x)$ then there is $x_o \in A_{m,n}(x)$ such that $x_o \in \text{int}(S(y,m)) \subset S(y,m)$, i.e., $y \in S(x_o,m)$. But $S(x_o,m) \subset S(x,n)$, implies $y \in S(x,n)$. Hence $A^-_{m,n}(x) \subset S(x,n)$ for fixed m and n .

Proposition 4.1.3 If (x,d) is a semi-metric space, then X has a σ-cushioned paired network.

Proof. For each $x \in X$ and any positive integer n let us define

$$\underset{\sim}{V}_n = \{S(x, \tfrac{1}{n}) \mid x \in X\} \quad ,$$

$$S^{\#}(x; \tfrac{1}{n} , \tfrac{1}{m}) = \{z \in X \mid S(x, \tfrac{1}{m}) \subset S(x, \tfrac{1}{n})\}$$

and finally

$$V_{n,m} = \{(S^{\#}(x; \frac{1}{n}, \frac{1}{m}), S(x, \frac{1}{n})) \mid x \in X\}$$

for $m \geqq n$. We shall show that for each fixed n and arbitrary m the collection $\underset{\sim}{V}_{n,m}$ is cushioned; i.e., we want to show that for arbitrary $A \subset X$,

$$\overline{\cup(S^{\#}(x; \frac{1}{n}, \frac{1}{m}) \mid x \in A)} \subset \cup(S(x, \frac{1}{n}) \mid x \in A) .$$

Let $t \in \overline{\cup(S^{\#}(x; \frac{1}{n}, \frac{1}{m}) \mid x \in A)}$. Then $S(t, \frac{1}{m})$ has non-empty intersection with $M_{A;n,m} = \cup(S^{\#}(x; \frac{1}{n}, \frac{1}{m}) \mid x \in A)$. Now if $x_o \in M_{A;n,m} \cap S(t, \frac{1}{m})$, then $t \in S(x_o, \frac{1}{m})$, and since $S(x_o, \frac{1}{m}) \subset S(x, \frac{1}{n})$, we find that $t \in \cup(S(x, \frac{1}{n}))$. Since A is an arbitrary subset of X, it is proved that $\underset{\sim}{V}_{n,m}$ is a cushioned collection for each fixed n and $m \geqq n$. Finally, we shall show that $V = \bigcup_{n=1}^{\infty} \bigcup_{m \geqq n}^{\infty} \underset{\sim}{V}_{n,m}$ is a paired network for X.

Let U be an open subset of X containing x. Since (X,d) is a semi-metric space, there is n_x such that $x \in S(x, \frac{1}{n_x}) \subset U$. Now, for each $m \geqq n_x$ we have $x \in S^{\#}(x; \frac{1}{n_x}, \frac{1}{m}) \subset S(x, \frac{1}{n_x}) \subset U$. Hence the proposition is proved.

The proof of the next Theorem 4.1.4 follows exactly on the same lines as Theorem 4.5 of Arhangelskii [4]. But for the sake of completeness, we give the proof.

Theorem 4.1.4 A regular space X is a semi-metric space iff it is first countable and has a σ-cushioned paired network.

Proof. Necessity of the conditions is obvious from Proposition 4.1.1 and Proposition 4.1.3.

We shall show that the conditions are sufficient. Let $\{V_n(x)\}_{n=1}^{\infty}$ be an open countable neighborhood base at x ; without loss of generality we assume that $V_n(x) \subset V_m(x)$ for all $n > m$. Let $\underset{\sim}{W} = \bigcup_{n=1}^{\infty} \underset{\sim}{W}_n$ be a σ-cushioned paired network for X , where $\underset{\sim}{W}_n = \{(W^n_{\alpha 1}, W^n_{\alpha 2}) \mid \alpha \in \Lambda_n\}$ for each n , is a paired cushioned collection. Let us now define

$$M^k_x = \overline{\cup (W^n_{\alpha 1} \mid n \leqq k, \ \alpha \in \Lambda_n \ \text{ and } \ x \notin W^n_{\alpha 2})}$$

$$U_k(x) = V_k(x) - M^k_x \qquad (k = 1,2,\ldots)$$

$$N(x,y) = \max \{n : (U_n(x) \cap U_n(y)) \cap (\{x\} \cup \{y\}) \neq \phi\}$$

and

$$d(x,y) = \frac{1}{N(x,y)} \quad \text{for all } x,y \text{ in } X .$$

It is obvious, that $d(x,y) = d(y,x)$. If $x = y$, then $N(x,y) = \infty$ and so $d(x,y) = 0$. If $d(x,y) = 0$, then either $y \in U_n(x) \subset V_n(x)$ for an infinite set of values of n , or vice-versa. This is possible only if $x = y$.

We now show that d is a semi-metric for X . Let M be a subset of X and x_o be any point of X such that $x_o \notin M^-$. By

the regularity of X there is n_o such that $V_{n_o}(x_o) \cap M^- = \phi$. Now let us choose n_1 such that $x_o \in W_{\alpha_1}^{n_1} \subset W_{\alpha_2}^{n_1} \subset X - M^-$. We claim that $d(x_o, M^-) \geqq \frac{1}{\max(n_1, n_o)}$; i.e. for any $y_o \in M^-$, $N(x_o, y_o) \leqq \max(n_1, n_o)$. We know that $U_{n_o}(x_o) \subset V_{n_o}(x_o) \subset X - M^- \subset X - y_o$, and on the otherhand $U_{n_1}(y_o) \subset V_{n_1}(y_o) - M_{y_o}^{n_1} \subset V_{n_1}(y_o) - W_{\alpha_1}^{n_1} \subset V_{n_1}(y_o) - x_o$. Hence for $n^1 > \max(n_o, n_1)$, $(U_{n^1}(y_o) \cap U_{n^1}(x_o)) \cap (\{x_o\} \cup \{y_o\}) \subset (U_{n_o}(x_o) \cap U_{n_1}(y_o)) \cap (\{x_o\} \cup \{y_o\}) = \phi$, i.e.,

$N(x_o, y_o) \leqq \max(n_o, n_1)$ which proves our claim that $d(x_o, M^-) > 0$.

Suppose M is a subset of X and x_o is a point of X such that $d(x_o, M) > 0$. We claim that $x_o \notin M^-$. Let $d(x_o, M) = \varepsilon_o$. Let us choose n_o a positive integer such that $n_o > \frac{1}{\varepsilon_o}$. The set $M_{x_o}^{n_o}$ is a closed set not containing x_o . Consequently, by the regularity of X there is a neighborhood $V_{n_1}(x_o)$ of X_o disjoint from $M_{x_o}^{n_o}$. Then for $n^1 > \max(n_o, n_1)$ we have $V_{n^1}(x_o) \subset V_{n_o}(x_o) - M_{x_o}^{n_o} = U_{n_o}(x_o)$. But if $y \in U_{n_o}(x_o)$ then $N(x_o, y) \geqq n_o$, and $d(x_o, y) = \frac{1}{N(x_o, y)} \leqq \frac{1}{n_o} < \varepsilon_o$ so that $U_{n_o}(x_o) \cap M = \phi$. Hence d is a semi-metric for X .

Remark 4.1.2 Recently, Theorem 4.1.4 has been stated in Kofner [31].

Proposition 4.1.5 In a T_2 semi-metric space X every closed set

is a G_δ set and X has a G_δ-diagonal.

<u>Proof</u>. Follows from Proposition 4.1.3, Proposition 2.1.2 and Corollary 2.1.3A.

<u>Proposition 4.1.6</u> If X is a space with σ-cushioned paired network then it is F_σ-screenable.

<u>Proof</u>. Let $\underset{\sim}{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover of X and let Λ be well ordered by $<$. Let $\underset{\sim}{W} = \bigcup_{i=1}^{\infty} \underset{\sim}{W}_i$ be a σ-cushioned paired network for X , where $\underset{\sim}{W} = \{(W^n_{\alpha 1}, W^n_{\alpha 2}) \mid \alpha \in \Lambda_n\}$ for each n is a paired cushioned collection. Let

$$h(U_{\alpha,1}) = \overline{\cup(W^i_{\alpha 1} \mid W^i_{\alpha 2} \subset U_\alpha)} \quad \text{for} \quad \alpha \in \Lambda_i \quad \text{and} \quad i = 1,2,\ldots .$$

Further for each i , define $W_{1i} = h(U_{1,i})$ and for $\alpha > 1$, $V_{\alpha i} = h(U_{\alpha,i}) - \cup(U_\beta \mid \beta \in \Lambda, \beta < \alpha)$. For each $\alpha \in \Lambda$ and each i , $V_{\alpha i}$ is a closed subset of X . Let $x \in X$. Let γ be the first member of Λ such that $x \in U_\gamma$. If $\beta > \gamma$, then $U_\gamma \cap V_{\beta i} = \phi$. If $\beta < \gamma$, then $V_{\beta i} \subset h(U_{\beta,i}) \subset h((U_\alpha \mid \alpha \in \Lambda, \alpha < \gamma),i) \subset \cup(U_\alpha \mid \alpha \in I, \alpha < \gamma)$. Since $\cup(U_\alpha \mid \alpha \in \Lambda, \alpha < \gamma)$ does not contain x , $U_\gamma \cap [X - h(\cup(U_\alpha \mid \alpha \in \Lambda, \alpha < \gamma),i)]$ is an open subset of X containing x which intersects $V_{\alpha i}$, only if $\alpha = \gamma$. Therefore $\underset{\sim}{V}_i = \{V_{\alpha i} \mid \alpha \in \Lambda\}$ is a discrete collection of closed sets. Now we need to show that $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$ is a cover of X . Since $\underset{\sim}{W}$ is a σ-cushioned paired network there is k such that $x \in h(U_\gamma,k)$ and therefore, $x \in V_{\gamma k} = h(U_\gamma,k) - \cup(U_\alpha \mid \alpha \in \Lambda, \alpha < \gamma)$ and $\underset{\sim}{V}$ is a

cover of X . $\underset{\sim}{V}$ is a refinement of $\underset{\sim}{U}$ is obvious. Hence X is F_σ-screenable.

An immediate consequence of Proposition 4.1.6 is that every semi-metric space is F_σ-screenable. This was first proved by McAuley [34].

<u>Theorem 4.1.7</u> A T_1-space X is semi-metric iff there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of symmetric subsets of $X \times X$ satisfying"

(a) $\bigcap_{i=1}^{\infty} V_i = \Delta$;

(b) for $x \in X$, $\{V_i[x]\}_{i=1}^{\infty}$ forms a base for the neighborhood system at x .

<u>Proof</u>. Let (X,d) be a semi-metric space. Then let us define by $V_i = \{(x,y) \in X \times X \mid d(x,y) < \frac{1}{i}\}$ for each i . Now it is easy see that the countable family $\{V_i\}_{i=1}^{\infty}$ of subsets of $X \times X$ has the required properties.

Conversely, let us assume that there exists a countable family $\{V_i\}_{i=1}^{\infty}$ of symmetric subsets of $X \times X$ satisfying the required properties. Let us assume that $V_{i+1} \subset V_i$ for each i . Now we define a real valued function $d : X \times X \to R$ by setting:

$$d(x,y) = 0 \quad \text{iff} \quad (x,y) \in V_i \quad \text{for all } i ;$$

$$d(x,y) = 1 \quad \text{iff} \quad (x,y) \notin V_i \quad \text{for all } i ;$$

and

$$d(x,y) = \frac{1}{i+1} \quad \text{iff} \quad (x,y) \in V_i - V_{i+1} .$$

It is easy to see that $d(x,y) = d(y,x)$ and $d(x,y) = 0$ iff $x = y$. Suppose C is a closed subset of X and $x_o \in X - C$. Then by (b) there is n_o such that $V_{n_o}[x_o] \subset X - C$. Now for each $y \in C$, it is easy to see that $d(x_o,y) \geqq \frac{1}{n_o}$, for if $d(x_o,y) < \frac{1}{n_o}$. Then, by the definition of d we have $y \in V_{n_o}[x_o]$ a contradiction, i.e., if C is closed and $x_o \notin C$ then $d(x_o,C) \neq 0$. Furthermore, suppose that C is a subset of X and $d(x_o,C) = \varepsilon_o \neq 0$. Let n be a positive integer such that $\varepsilon_o > \frac{1}{n}$. Then, it is easy to verify that $V_n[x_o] \cap C = \phi$. Consequently, $x_o \notin C^-$. Hence d is a semi-metric for X which proves the theorem.

In addition C.C. Alexander [2] announced the following interesting theorem.

<u>Theorem 4.1.8</u> Let X be a T_1-space. Then X is a semi-metric space iff there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of covers (not necessarily open) of X such that the following conditions are satisfied.

(a) $\underset{\sim}{V}_{i+1}$ refines $\underset{\sim}{V}_i$ for each i;

(b) for each $x \in X$, $\{St(x,\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ is a base for the neighborhood system at x.

<u>Proof</u>. Is obvious.

Theorem 4.1.8 suggests the following definition:

Definition 4.1.2 A topological space X is a semi-strict p-space iff there exists a countable family of covers $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ (not necessarily open) of X satisfying:

(i) $A_x = \bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i)$ is a compact set for $x \in X$;

(ii) the family $\{St(x,\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ is a base for the open sets containing A_x for each $x \in X$.

Theorem 4.1.9 A T_2-space X is a semi-metric space iff it is a semi-strict p-space and has a $\bar{G_\delta}$-diagonal.

Proof. Necessity follows from Proposition 3.3.15 and Theorem 4.1.8.

We now prove the sufficiency part. Since X has a $\bar{G_\delta}$-diagonal there exists a countable family $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ of open covers of X such that for each $x \in x$, $\bigcap_{i=1}^{\infty} \overline{St(x,\underset{\sim}{W}_i)} = \{x\}$. Also X a semi-strict p-space, so there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of covers of X satisfying conditions (i) and (ii) of Definition 4.1.2. Without loss of generality we may assume that $\underset{\sim}{W}_{i+1}$ refines $\underset{\sim}{W}_i$ and $\underset{\sim}{V}_{i+1}$ refines $\underset{\sim}{V}_i$ for each $i = 1,2,\dots$. Let us define $\underset{\sim}{U}_i = \underset{\sim}{V}_i \wedge \underset{\sim}{W}_i = \{V \cap W \mid V \in \underset{\sim}{V}_i$ and $W \in \underset{\sim}{W}_i\}$ for each i .

Now it is easy to see that the countable family $\{\underset{\sim}{U}_i\}_{i=1}^{\infty}$ of covers of X satisfies the conditions (a) and (b) of Theorem 4.1.8. Hence X is semi-metrizable.

Definition 4.2.1 A topological space X is said to be point countable type iff each $x \in X$ is contained in a compact set of

countable character.

Theorem 4.2.1 Let $f : X \to Y$ be a continuous and closed mapping of a regular semi-metric space X onto a completely regular space Y. Then Y is a semi-metric space iff Y is point countable type.

Proof. Let $f : X \to Y$ be a continuous closed mapping of a semi-metric space X onto a completely regular space Y which is point countable type. By Proposition 4.1.3 and Proposition 2.1.8, Y has σ-cushioned paired network. Now in view of Theorem 4.1.4, we need only show that Y is first countable. Let $y \in Y$ be any point, then by Proposition 2.2.1, there exists open sets $\{V_y^i\}_{i=1}^{\infty}$ such that $y = \bigcap_{i=1}^{\infty} V_y^i$ and $V_y^{i+1} \subset V_y^i$. Since Y is point countable type, there is a compact set F_y such that $y \in F_y$ and F_y is of countable character. Let $Q_i = V_y^i \cap F_y$. The sets Q_i, $i = 1,2,\ldots$, are open in F_y and $\bigcap_{i=1}^{\infty} Q_i = \{y\}$. But F_y is compact implies y is of countable character in F_y. Now y is of countable character in F_y and F_y is of countable character in Y. Applying Proposition 3.3 of Arhangel'skii [5] y is of countable character in Y. Since y is an arbitrary point of Y. We have shown that Y is first countable. Hence by Theorem 4.1.4, Y is a semi-metric space.

The converse follows from the fact that every first countable space is point countable type.

Proposition 4.2.2 Let X be a normal σ-paracompact space, Y a locally

compact (or first countable) T_1-space and $f : X \to Y$ continuous and closed. Then the boundary of $f^{-1}y$, denoted by $\partial f^{-1}y$, is compact.

Proof. By Theorem 2.1 and Corollary 2.2 of Michael [39] $\partial f^{-1}y$ is countably compact. Since $\partial f^{-1}y$ is a closed subset of X and X is σ-paracompact, $\partial f^{-1}y$ is σ-paracompact by Corollary 3.1.6B. Finally, $\partial f^{-1}y$ is compact by Corollary 3.3.11A. Hence the proposition is proved.

In view of the preceding results, the following makes an attractive conjecture.

Conjecture. Let f be a closed continuous mapping of a normal semi-metric space X onto a regular space Y. Then the following are equivalent:

(a) Y is first countable;

(b) for each $y \in Y$, $\partial f^{-1}y$ is compact;

(c) Y is a semi-metric space.

Proposition 4.2.4 Let X be a regular topological space and $f : X \to Y$ be a perfect mapping of X onto a semi-metric space Y. Then X is a σ-paracompact semi-strict p-space.

Proof. In view of Theorem 3.2.7 and Corollary 3.2.1A, it is enough to show that X is a semi-strict p-space.

Since Y is a semi-strict p-space by Theorem 4.1.8 there

exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of covers of Y such that $\{St(y_1,\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ is a base for the neighborhood system at y. Let us define $\underset{\sim}{W}_i = \{f^{-1}V \mid V \in \underset{\sim}{V}_i\}$ for $i = 1,2,\ldots$. Now for each $x \in X$, it is easy to see that $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{W}_i) = f^{-1}y$ where $y = fx$. Now it remains to show that $\{St(x,\underset{\sim}{W}_i)\}_{i-1}^{\infty}$ is a base for the neighborhood of $f^{-1}y$. Let U be an open set containing $f^{-1}y$. Since f is closed $O = Y - f(X - U)$ is an open set in Y containing y and such that $f^{-1}O \subset U$. Because $\{St(y,\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ is a base for the neighborhood system at y we have $St(y,\underset{\sim}{V}_i) \subset O$ for some i. Now by the continuity of f, $f^{-1}(St(y,\underset{\sim}{V}_i))$ is a neighborhood of $f^{-1}y$ contained in $f^{-1}O \subset U$. But $f^{-1}S(y,\underset{\sim}{V}_i) = St(x,\underset{\sim}{W}_i)$ for $x \in f^{-1}y$; so $\{St(x,\underset{\sim}{W}_i)\}_{i=1}^{\infty}$ is a base for the open sets containing $f^{-1}y$ where $y = fx$. Hence the proposition is proved.

<u>Definition 4.2.2</u> A continuous mapping $f : X \to Y$ is called a <u>weak W-mapping</u> for an open cover W of X iff for each point $y \in Y$, $f^{-1}y$ is contained in some member of W.

<u>Theorem 4.2.4</u> In order that a space X be contractible to a Hausdorff semi-metric space, it is necessary and sufficient that there exists a countable family of open covers $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of X satisfying the following conditions:

(a) $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i) = \{x\}$ for $x \in X$;

(b) for every $\underset{\sim}{V}_i$ there exists a weak $\underset{\sim}{V}_i$-mapping $f_i : X \to R_i$ on X into a Hausdorff semi-metric space R_i.

Proof. The necessity of the conditions is obvious.

For sufficiency, let us define a function $f : X \to f(x) \subset \prod_{i=1}^{\infty} R_i$, where $f(x) = \{f_i(x)\}_{i=1}^{\infty} \in \prod_{i=1}^{\infty} R_i$. Clearly, f is continuous and $f(X)$ being a subset of a semi-metric space $\prod_{i=1}^{\infty} R_i$ is a semi-metric space. We need only show that f is one to one. Let $f(x) = f(y)$. Then, $f_i(x) = f_i(y)$, for each i so that $y \in St(x, \underset{\sim}{V}_i)$, for each i. But $\bigcap_{i=1}^{\infty} St(x, \underset{\sim}{V}_i) = \{x\}$, so $x = y$. Hence $f(x) = f(y)$ iff $x = y$; i.e., f is one to one. Thus the theorem is proved.

CHAPTER V

QUASI-METRIC SPACES

The notion of quasi-metric was introduced by Wilson [57] and has also been studied by Albert [1], Ribeiro [49], and more recently by Kelly [29], Pervin [48], Stoltenberg [52], Patty [47] and Sion and Zelmer [50]. In this chapter, we extend and unify some of their work.

In Section 1 we give some simple properties possessed by quasi-metric spaces.

In Section 2 we give some theorems characterizing quasi-metrizability of a topological space.

<u>Definition 5.1.1</u> A <u>quasi-metric</u> for a set X is a non-negative real valued function d on $X \times X$ such that for x,y,z in X,

(1) $d(x,y) = 0$ iff $x = y$;

(2) $d(x,y) \leqq d(x,z) + d(z,y)$.

Let d be a quasi-metric on a set X, and define $d^1 : X \times X \to R$ by the equation $d^1(x,y) = d(y,x)$. It is easy to verify that d^1 is a quasi-metric on X . d^1 and d are called <u>conjugate quasi-metrics</u> on X . The <u>quasi-metric topology</u> on X is the family of all sets U in X such that for each $x \in U$ there

is $\varepsilon > 0$ with the property that $S(x,\varepsilon) = \{y \mid d(x,y) < \varepsilon\} \subset U$. The quasi-metric topology will be denoted by τ_d and X with topology τ_d is called a <u>quasi-metric space</u>. X with τ_{d^1} where d^1 is a conjugate quasi-metric of d is called the <u>conjugate quasi-metric space</u>.

<u>Proposition 5.1.1</u> Let X be a set with quasi-metric d, A be a subset of X and x,y be any pair of points of X. Then $d(A,y) - d(A,x) \leqq d(x,y)$, where $d(A,y) = \inf\{d(a,y) \mid a \in A\}$.

<u>Proof</u>. $d(A,y) \leqq d(a,y)$ for all $a \in A$. Now, by the triangle inequality $d(A,y) \leqq d(a,y) \leqq d(a,x) + d(x,y)$ for all $a \in A$. Hence, it follows that $d(A,y) \leqq d(A,x) + d(x,y)$; i.e., $d(A,y) - d(A,x) \leqq d(x,y)$. Hence the proposition is proved.

<u>Proposition 5.1.2</u> Let (X,τ_d) be a quasi-metric space. Then, every closed subset of X is a G_δ set in (X,τ_{d^1}).

<u>Proof</u>. Let C be a closed subset of X with respect to the topology τ_d. Let $U_i = \{x \in X \mid d^1(C,x) < \frac{1}{i}\}$ for $i = 1,2,\ldots$. We first show that U_i is open with respect to τ_{d^1}. Let $x \in U_i$ and $r = \frac{1}{i} - d^1(C,x)$. Let $y \in S^1(x,r) = \{z \in X \mid d^1(x,z) < r\}$. Then, we have $d^1(x,y) < \frac{1}{i} - d^1(C,x)$. Therefore, by Proposition 5.1.1, we have $d^1(C,y) - d^1(C,x) < \frac{1}{i} - d(C,x)$, i.e., $d^1(C,y) < \frac{1}{i}$, so that $y \in U_i$. It is easy to see that each U_i is open with respect to τ_{d^1}. It remains then to show that $C = \bigcap_{i=1}^{\infty} U_i$. It is easy to show that $C \subset U_i$ for $i = 1,2,\ldots$, i.e., $C \subset \bigcap_{i=1}^{\infty} U_i$. Let $y \in \bigcap_{i=1}^{\infty} U_i$. Then $d^1(C,y) < \frac{1}{i}$ for all $i = 1,2,\ldots$, i.e.,

$d(y,C) < \frac{1}{i}$ for all $i = 1,2,\ldots$. Since $d(y,C) < \frac{1}{i}$ for each i there is $x_i \in C$ such that $d(y,x_i) < \frac{1}{i}$, i.e., the sequence $\{x_i\}_{i=1}^{\infty}$ converges to y in τ_d . But C is closed in τ_d , so that $y \in C$. Since y is an arbitrary member of $\bigcap_{i=1}^{\infty} U_i$ implies $\bigcap_{i=1}^{\infty} U_i \subset C$. Therefore $C = \bigcap_{i=1}^{\infty} U_i$ where U_i is open with respect to τ_{d^1} for each i . Consequently, C is a G_δ set with respect to τ_{d^1} . Hence the proposition is proved.

<u>Remark 5.1.1</u> Let d be a quasi-metric for a set X . For each $m > 0$ define $\rho_m(x,y) = \min\{m,d(x,y)\}$. Then ρ_m is a quasi-metric for X and $\tau_d = \tau_{\rho_m}$. Thus any quasi-metric is equivalent to a bounded quasi-metric.

<u>Proposition 5.1.3</u> Let $\{(X_n,d_n)\}_{n=1}^{\infty}$ be a countable family of quasi-metric spaces, each of diameter at most one. For x and y belong to $\prod_{n=1}^{\infty} X_n$, define

$$d(x,y) = \sum_{n=1}^{\infty} \frac{d_n(x_n,y_n)}{2^n} .$$

Then d is a quasi-metric for $\prod_{n=1}^{\infty} X_n$, and the resulting quasi-metric topology is the product topology. Also d^1 defined by

$$d^1(x,y) = \sum_{n=1}^{\infty} \frac{d^1(x_n,y_n)}{2^n}$$

is the conjugate quasi-metric for $\prod_{n=1}^{\infty} X_n$ and its quasi-metric topology is the product of the topologies generated by the conjugate quasi-metrics d_n^1 , $n = 1,2,\ldots$.

Proof. Similar to Theorem 14 on page 122 of Kelley [28].

Proposition 5.1.4 If (X,d) is a regular quasi-metric space, then (X,d^1) is a Hausdorff quasi-metric space.

Proof. See Corollary 3.19 on page 43 of Murdeshwar and Naimpally [42].

Corollary 5.1.4A Let (X,d) be a regular quasi-metric space. Then $\Delta = \{(x,x) \mid x \in X\}$ is a G_δ set in $X \times X$ with respect to $\tau_d \times \tau_d$.

Proof. Since (X,d) is a regular quasi-metric space, by Proposition 5.1.4 (X,d^1) is a Hausdorff quasi-metric space. Therefore Δ is a closed subset of $X \times X$ in $\tau_{d^1} \times \tau_{d^1}$. Now, by Proposition 5.1.2 it is easy to see that Δ is a G_δ subset of $X \times X$ with respect to the topology $\tau_d \times \tau_d$.

Theorem 5.2.1 [Ribeiro [49]) A T_1-space X is quasi-metrizable iff for each $x \in X$ there is a base for the neighborhood system $\{U(x,n)\}_{n=o}^{\infty}$ such that $U(x,o) = X$ and if $y \in U(x,n)$ for $n > 0$, then $U(y,n) \subset U(x,n-1)$.

Proof. Let (X,d) be a quasi-metric space. Let us define $U(x,n) = S(x, \frac{1}{2^n})$, where for each $x \in X$ and positive integer $n > 0$, $S(x, \frac{1}{2^n}) = \{y \mid d(x,y) < \frac{1}{2^n}\}$ and $U(x,o) = S(x,o) = X$. Now, obviously $\{U(x,n)\}_{n=o}^{\infty}$ is a countable base for the neighborhood system at x

and, if $y \in U(x,n)$ and $z \in U(y,n)$, then

$$d(x,z) \leqq d(x,y) + d(y,z) < \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}} \quad ;$$

i.e., $z \in U(y,n)$ implies $z \in U(x,n-1)$. Thus, for any $y \in U(x,n)$ we have $U(y,n) \subset U(x,n-1)$. Consequently, $\{U(x,n)\}_{n=o}^{\infty}$ is a required countable base for the neighborhood system at x.

Conversely, for each $x \in X$, let $\{U(x,n)\}_{n=o}^{\infty}$ be a base for the neighborhood system with the property described in the hypothesis of the theorem. Let us define

$$V_n = \cup(\{x\} \times U(x,n) \mid x \in X) \quad \text{for} \quad n = 1,2,\ldots \quad .$$

It is easy to see that $\{V_n\}_{n=1}^{\infty}$ is a countable base for a quasi-uniformity. We verify only the triangle inequality, i.e., we want to show that $V_n \circ V_n \subset V_{n-1}$ for each $n = 1,2,\ldots$ where $V_o = X \times X$. Let $(x,y) \in V_n \circ V_n$. Then there is z such that (x,z) and $(z,y) \in V_n$. Thus $z \in U(x,n)$ and $y \in U(z,n)$; also $U(z,n) \subset U(x,n-1)$, so that $y \in U(x,n-1)$, i.e., $(x,y) \in \{x\} \times U(x,n-1) \subset V_{n-1}$. Hence by Theorem 11.1.1 on page 175 of Pervin [45] X is quasi-metrizable.

<u>Theorem 5.2.2</u> A T_1-space X is quasi-metrizable iff for each $x \in X$ there exists a decreasing base $\{U_n(x)\}_{n=1}^{\infty}$ for the neighborhood system such that for each n there exist neighborhoods $S_n^1(x)$, $S_n^2(x)$ of x with the property that $y \in S_n^1(x)$ implies $S_n^2(y) \subseteq U_n(x)$ and $U_{n+1}(x) \subset S_n^1(x) \cap S_n^2(x)$.

Proof. Suppose X is a quasi-metric space. Then by the Theorem 5.2.1, for each $x \in X$ there exists a countable base $\{U(x,n)\}_{n=o}^{\infty}$ for the neighborhood system such that if $y \in U(x,n)$ for $n = 1,2,\ldots$, then $U(y,n) \subset U(x,n-1)$ where $U(x,o) = X$. Let us define for each $n > 0$, $U_n(x) = U(x,n)$ and $S_n^1(x) = S_n^2(x) = U(x,n+1)$. Now, it is easy to verify that $\{U_n(x)\}_{n=o}^{\infty}$ where $U_o(x) = X$ and for each n , $S_n^1(x) = S_n^2(x)$ has the required properties.

Conversely, let us suppose that for each $x \in X$ there exists a decreasing countable base $\{U_n(x)\}_{n=1}^{\infty}$ for the neighborhood system at x and for each n there exist neighborhoods $S_n^1(x)$, $S_n^2(x)$ such that $y \in S_n^1(x)$ implies $S_n^2(y) \subset U_n(x)$ and $U_{n+1}(x) \subset S_n^1(x) \cap S_n^2(x)$. Let us define $U(x,o) = X$ and $U(x,n) = S_n^1(x) \cap S_n^2(x)$ for $n > 0$. Clearly $\{U(x,n)\}_{n=o}^{\infty}$ is a countable base for the neighborhood system at x . We now need only show that if $y \in U(x,n)$ then $U(y,n)$ $U(x,n-1)$. If $y \in U(x,n) = S_n^1(x) \cap S_n^2(x)$ then $S_n^1(y) \cap S_n^2(y) = U(y,n) \subset S_n^2(y) \subset U_n(x) \subset S_{n-1}^1(x) \cap S_{n-2}^2(x) = U(x,n-1)$; i.e., if $y \in U(x,n)$ then $U(y,n) \subset U(x,n-1)$. Hence the theorem is proved.

Corollary 5.2.2A A T_1-space X is quasi-metrizable iff there exist sequences of neighborhoods of X , $\{U_n(x)\}_{n=1}^{\infty}$ and $\{S_n(x)\}_{n=1}^{\infty}$, such that:

(a) for each $x \in X$, $\{U_n(x)\}_{n=1}^{\infty}$ is a base for the neighborhood ystem at x ;

(b) for all $x,y \in X$, $y \in S_n(x)$ we have $S_n(y) \subset U_n(x)$ and $U_{n+1}(x) \subset S_n(x)$ for all $n = 1,2,\ldots$.

Theorem 5.2.3 A T_1-space X is quasi-metrizable iff there is a collection $\{U_n(x)\}_{n=o}^{\infty}$ of open sets such that:

(a) for all $x \in X$, $\{U_n(x)\}_{n=o}^{\infty}$ is a decreasing base for the neighborhood system at x where $U_o(x) = X$, and

(b) for $y \in X$ and $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are sequences such that $z_n \in U_n(y_n)$ and $y_n \in U_n(y)$ for each n, then $\{z_n\}_{n=1}^{\infty}$ converges to y and $z_n \in U_{n-1}(y)$ for all $n = 1,2,\ldots$.

Proof. Let X be a quasi-metric space with a quasi-metric d. Let $U_o(x) = X$ and for each $x \in X$ and $n > 0$, $U_n(x) = S(x, \frac{1}{2^n}) = \{y \mid d(x,y) < \frac{1}{2^n}\}$. Clearly, $\{U_n(x)\}_{n=o}^{\infty}$ is a base for the neighborhood system at x. Now, if y is an arbitrary point of X and $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are sequences such that $z_n \in U_n(y_n)$ and $y_n \in U_n(y)$ then $d(y,z_n) \leqq d(y,y_n) + d(y_n z_n) < \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}$. Now, it is easy to see that $z_n \in U_{n-1}(y)$ for all n and $\{z_n\}_{n=1}^{\infty}$ converges to y .

Conversely, suppose that there exists a collection $\{U_n(x)\}_{n=o}^{\infty}$ of open sets such that, it is a base for the neighborhood system at x, where $U_o(x) = X$, for any $x \in X$. If $y \in X$ and $\{y_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are sequences such that $z_n \in U_n(y_n)$ and $y_n \in U_n(y)$ for each n then $\{z_n\}_{n=1}^{\infty}$ converges to y and $z_n \in U_{n-1}(y)$ for all n. We claim that the collection $\{U(x,o)\}_{n=o}^{\infty}$ where $U(x,o) = X$ and $U(x,n) = U_n(x)$ for $n = 1,2,\ldots,$ has the property that if $y \in U(x,n)$ for $n > 0$ then $U(y,n) \subset U(x,n-1)$. Suppose for some

$x_o \in X$ and some n_o there is $y_{n_o} \in U(x_o,n_o)$ such that $U(U_{n_o},n_o) \not\subset U(x_o,n_o-1)$. Let us choose $z_{n_o} \in U(y_{n_o},n_o) - U(x_o,n_o-1)$ and for $n \neq n_o$ choose $z_n \in U_n(y_n)$ where $y_n \in U_n(x)$. Clearly, $\{z_n\}_{n=1}^{\infty}$ converges to x_o but $z_{n_o} \notin U(x_o,n_o-1)$, a contradiction. Hence the theorem follows from Theorem 5.2.1.

Theorem 5.2.4 Let X be a T_1-space such that there exists a countable family $\{\underset{\sim}{B}_i\}_{i=1}^{\infty}$ of open covers of X satisfying the following conditions:

(a) $\underset{\sim}{B}_{i+1}$ refines $\underset{\sim}{B}_i$ for each i and $\underset{\sim}{B}_i \subset \underset{\sim}{B}_{i+1}$;

(b) for each i and each $x \in X$, $\cap(B | x \in B \in \underset{\sim}{B}_i) = 0_x^i$ is a neighborhood at x and $\{0_x^i\}_{i=1}^{\infty}$ is a base for the neighborhood system at x for all $x \in X$. Then X is quasi-metrizable.

Proof. For each $x \in X$ and i define $U(x,o) = X$ and $U(x,i) = 0_x^i$ for $i > 0$. By (a) $U(x,i+1) \subset U(x,i)$ for all i and by (b) $\{U(x,i)\}_{i=o}^{\infty}$ is a base for the neighborhood system at x. Also, if $y \in U(x,i)$ by the construction it is easy to see that $U(y,i) \subset U(x,i)$. Hence by Theorem 5.2.1, X is quasi-metrizable.

Theorem 5.2.5 A T_1-space with σ-point finite base is quasi-metrizable.

Proof. Let X be a T_1-space with σ-point finite base $\underset{\sim}{B} = \bigcup_{i=o}^{\infty} \underset{\sim}{B}_i$, $\underset{\sim}{B}_o = X$ and $\underset{\sim}{B}_i$ is point finite for each i. Without loss of

generality let us assume that $\underset{\sim}{B}_i \subset \underset{\sim}{B}_{i+1}$ for each i. Now for each $x \in X$ define $U(x,o) = X$ and $U(x,i) = \cap(B \mid x \in B \in \underset{\sim}{B}_i)$. Since $\underset{\sim}{B}_i$ is point finite, $U(x,i)$ is a finite intersection of open sets and therefore open. Also, since $\underset{\sim}{B}$ is a base, $\{U(x,i)\}_{i=o}^{\infty}$ is a base for the neighborhood system at $x \in X$. Furthermore if $y \in U(x,i)$ for $i > 0$ by the construction of $U(x,i)$, it is easy to see that $U(y,i) \subset U(x,i) \subset U(x,i-1)$. Hence by Theorem 5.2.1, X is quasi-metrizable.

Corollary 5.2.5B If a topological space X is such that there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of point finite open coverings satisfying the conditon:

(*) for any $x \in X$ there is a sequence $\{V_n\}_{n=1}^{\infty}$, where $V_n \in \underset{\sim}{V}_n$, forming a base for the neighborhood system at x, then X is quasi-metrizable.

Proof. Similar to Theorem 5.2.4.

Definition 5.2.1 A quasi-metric (X,d) is called a conjugate strong quasi-metric iff $\tau_d \subset \tau_{d^1}$.

Theorem 5.2.6 A quasi-metric space (x,d) is conjugate strong iff for any two sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ such that $d(x_n,x) \to 0$ and $d(x_n,y_n) \to 0$ it follows that $d(x,y_n) \to 0$.

Proof. Let $x \in X$ and $S(x, \frac{1}{2^{n_o}}) = \{y \mid d(x,y) < \frac{1}{2^{n_o}}\}$ be a basic

neighborhood of x and suppose that for all m,

$$S^1(x, \frac{1}{2^m}) = \{y \mid d^1(x,y) < \frac{1}{2^m}\}$$

is such that

$$S^1(x, \frac{1}{2^m}) - S(x, \frac{1}{2^{n_o}}) \neq \phi$$

where n_o is fixed. Let us choose $x_m \in S^1(x, \frac{1}{2^m}) - S(x, \frac{1}{2^{n_o}})$. Now choose $y_m = x_m$. Clearly $d(x_m,x) \to 0$ and $d(x_m,y_m) \to 0$ so that by the hypothesis $d(x,y_m) \to 0$. Hence for some $m > n_o$ we have $y_m \in S(x, \frac{1}{2^{n_o}})$, a contradition. Consequently, $S^1(x, \frac{1}{2^m}) \subset S(x, \frac{1}{2^m})$ for some $m > n_o$; i.e., $\tau_d \subset \tau_{d^1}$.

Conversely, if $\tau_d \subset \tau_{d^1}$ we want to show that for any pair of sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ for which $d(x_n,x) \to 0$ and $d(x_n,y_n) \to 0$, we have $d(x,y_n) \to 0$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences such that $d(x_n,x) \to 0$ and $d(x_n,y_n) \to 0$. Now $d(x_n,x) \to 0$ implies $d^1(x,x_n) \to 0$. Since $\tau_d \subset \tau_{d^1}$ we have $d(x,x_n) \to 0$. Using the triangle inequality the result follows immediately.

CHAPTER VI

DEVELOPABLE SPACES

The concept of developable spaces is quite old and has its origin in the works of R.L. Moore [40]. In recent years the problem of whether each normal developable space is metrizable has received a considerable amount of attention. However the problem still remains open.

In this chapter we list some familiar properties of developable spaces and an internal characterization is given in Theorem 6.1.2. We demonstrate the relation between semi-metric and developable spaces, quasi-metric and developable spaces. Also, we show that a locally developable F_σ-screenable space is a developable space.

<u>Definition 6.1.1</u> A topological space X is called <u>developable</u> iff there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X such that for any $x \in X$, $\{St(x,\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ is a base for the neighborhood system at x.

Clearly a developable space is σ-paracompact.

Let us recall several other properties of developable spaces which are either known or immediate consequences of the definition:

(a) Every developable space has a σ-discrete closed network. See Bing [11].

(b) Every developable T_1-space has G_δ-diagonal. See Theorem 2.3.1.

(c) Every completely regular developable space is a strict p-space. See Theorem 1.2.1.

(d) Every developable T_1-space is semi-metrizable. See Theorem 4.1.4.

(e) If X is a regular developable space, then there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X such that for x,y in X there is i_x such that $y \notin Cl_X(St(s,\underset{\sim}{V}_{i_x})$, i.e., X has $G_{\bar{\delta}}$-diagonal. See Definition 2.2.3.

<u>Theorem 6.1.1</u> A completely regular space X is a p-space and has G_δ-diagonal iff the diagonal of X is a closed G_δ set in $X \times BX$ where BX is any Hausdorff compactification of X .

<u>Proof</u>. Assuming X is a p-space, by Theorem 1.1.2 the diagonal $\Delta_X \subset \bigcap_{i=1}^{\infty} G_i \subset X \times X$, where for each i , G_i is open in $X \times BX$ and BX is a Hausdorff compactification of X . Also Δ_X is a G_δ set in $X \times X$ implies $\Delta_X = \bigcap_{i=1}^{\infty} H_i$ where H_i is open in $X \times X$ for each i . Let us choose U_i^1 open in $X \times BX$ such that $U_i^1 \cap (X \times X) = H_i$ and let $U_i = U_i^1 \cap G_i$ for each i . Because $\bigcap_{i=1}^{\infty} G_i \subset X \times X$ and $\Delta_X = \bigcap_{i=1}^{\infty} H_i$, it is easy to see that $\bigcap_{i=1}^{\infty} U_i = \Delta_X$ where U_i is open in $X \times BX$ for each i . Hence Δ_X is a G_δ set in $X \times BX$. Δ_X is closed in $X \times BX$ follows from the fact that BX is Hausdorff.

Conversely, let us assume that $\Delta_X = \bigcap_{i=1}^{\infty} U_i$ where for each

i , U_i is open in $X \times BX$ for BX a Hausdorff compactification of X . By Theorem 1.1.2, X is a p-space and $\Delta_X = \bigcap_{i=1}^{\infty} (U_i \cap (X \times X))$; therefore Δ_X is a G_δ set in $X \times X$. Hence the theorem is proved.

Corollary 6.1.1A A completely regular space X is a p-space and has G_δ-diagonal iff there exists a countable family $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X in BX such that $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i) = \{x\}$ for all $x \in X$, where BX is a Hausdorff compactification of X .

Proof. Is similar to Theorem 2.3.1.

Theorem 6.1.2 A completely regular space X is developable iff X is σ-paracompact and Δ_X is a closed G_δ set of $X \times BX$, where BX is a Hausdorff compactification of X .

Proof. If X is developable, it is trivially a σ-paracompact space and by Theorem 1.2.2 and Theorem 2.3.1, it is a p-space with G_δ-diagonal. Therefore, by Theorem 6.1.1, Δ_X is a closed G_δ set in $X \times BX$ where BX is a Hausdorff compactification of X .

Conversely, let X be a σ-paracompact space and Δ_X be a closed G_δ set in $X \times BX$. Since Δ_X is a closed G_δ set in $X \times BX$ by Corollary 6.1.1A there exists a countable $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of open covers of X in BX such that $\bigcap_{i=1}^{\infty} St(x,\underset{\sim}{V}_i) = \{x\}$ for all $x \in X$. For each i and $x \in X$ let us define O_x^i a neighborhood of x in BX such that $O_x^i \subset O_x^{i-1}$ and $Cl_{BX} O_x^i \subset V$ for some $V \in \underset{\sim}{V}_i$. Let $W_x^i = O_x^i \cap X$. Then for each i, $\underset{\sim}{W}_i = \{W_x^i \mid x \in X\}$ is an open cover

X . Now, since X is σ-paracompact, for each i there is a sequence $\{\underset{\sim}{W}^i_n\}_{n=1}^{\infty}$ of open covers of X satisfying the condition: for $x \in X$ there is n_x such that $St(x,\underset{\sim}{W}^i_{n_x}) \subset W$ for some $W \in \underset{\sim}{W}_i$. Without loss of generality we may assume that $\underset{\sim}{W}^i_{n+1}$ refines $\underset{\sim}{W}^i_n$ for each n . Finally, for any integer m let $\underset{\sim}{R}_m$ be an open cover of X such that $\underset{\sim}{R}_m$ refines each $\underset{\sim}{W}^t_s$ for $s \leqq m$, $t \leqq m$ and $\underset{\sim}{R}_{m+1}$ refines $\underset{\sim}{R}_m$. We shall show that the countable family $\{\underset{\sim}{R}_m\}_{m=1}^{\infty}$ of open covers of X is a development for X . If $x \in X$ is fixed and k is a positive integer, then there is some $x_k \in X$ and $n_k \geqq k$ such that $St(x,\underset{\sim}{W}^k_{n_k}) \subset 0^k_{x_k} \cap X$, so that $St(x,\underset{\sim}{R}_{n_k}) \subset St(x,\underset{\sim}{W}^k_{n_k}) \subset 0^k_{x_k} \cap X$. If $n_{k+1} > n_k$ we have $St(x,\underset{\sim}{R}_{n_{k+1}}) \subset St(x,\underset{\sim}{R}_{n_k})$. Suppose U is a neighborhood of x , open in X and $U^{\#}$ is an open set in BX such that $U = X \cap U^{\#}$. If $\bigcap_{k=1}^{m} Cl_{BX}0^k_{x_k} \not\subset U^{\#}$ for any m , then

$$\{\bigcap_{k=1}^{m} (Cl_{BX}0^k_{x_k} - U^{\#})\}_{m=1}^{\infty}$$

is a decreasing sequence of closed sets in BX and hence $\bigcap_{m=1}^{\infty} (\bigcap_{k=1}^{m} Cl_{BX}0^k_{x_k} - U^{\#}) \neq \phi$. Now

$$\bigcap_{k=1}^{\infty} Cl_{BX}0^k_{x_k} \subset [\bigcap_{k=1}^{\infty} St(x,\underset{\sim}{V}_k)] \cap [\bigcap_{k=1}^{\infty} W^k_{x_k}] \subset X \cap [\bigcap_{k=1}^{\infty} W^k_{x_k}] = \bigcap_{k=1}^{\infty} 0^k_{x_k} .$$

Hence $y \in \bigcap_{k=1}^{\infty} Cl_{BX}0^k_{x_k} - U^{\#}$ implies that $y \in \bigcap_{k=1}^{\infty} 0^k_{x_k}$ and then by Corollary 1.1.A it follows that $\{x_k\}_{k=1}^{\infty}$ converges to y , i.e., $y \in \bigcap_{i=1}^{\infty} St(x,V_i)$. Bur for any $x \in X$, $\bigcap_{i=1}^{\infty} St(x,V_i) = \{x\}$ so $x = y$ which is a contradiction. So there is m such that $\bigcap_{k=1}^{m} Cl_{BX}0^k_{x_k} \subset U^{\#}$. It follows that

$$St(x,\underset{\sim}{R}_{n_m}) \subset \bigcap_{k=1}^{m} St(x,\underset{\sim}{R}_{n_k}) \subset [\bigcap_{k=1}^{m} O^k_{x_k}] \cap X \subset U^{\#} \cap X = U \quad .$$

Hence $x \in St(x,\underset{\sim}{R}_{n_m}) \subset U$ and hence we have proved the claim. Thus the theorem is proved.

Corollary 6.1.2A A completely regular space is developable iff the following conditions are satisfied:

(i) X has a G_δ-diagonal;

(ii) X is σ-paracompact;

(iii) X is a p-space.

Proof. Follows from Theorems 6.1.1 and 6.1.2.

Theorem 6.1.4 A completely regular space X is developable iff X has a σ-cushioned paired network and is a p-space.

Proof. Necessity is trivial from the properties listed for developable space in the beginning. Sufficiency follows from Proposition 4.1.6, Corollary 3.2.1A, Proposition 2.1.2 and Theorem 6.1.2.

Corollary 6.1.4A A completely regular space X is developable iff X is a semi-metric and p-space.

Proof. Follows from Theorem 4.1.4, Theorem 1.2.2 and Theorem 6.1.4.

Corollary 6.1.4A answers the question raised by Morton Brown [15]:

"What is a topological condition which is necessary and sufficient for a semi-metric space to be developable"? Recently, several others have solved this problem independently.

Thorem 6.1.5 A F_σ-screenable locally developable space is a developable space.

Proof. Since X is locally developable for each $x \in X$ there is an open neighborhood W_x of x which is developable, i.e., there exists $\{\underset{\sim}{W}_n(x)\}_{n=1}^{\infty}$ a sequence of open coverings of W_x such that $\{St(y,\underset{\sim}{W}_n(x))\}_{n=1}^{\infty}$ is a base for the neighborhood system at y for $y \in W_x$. Further, X is F_σ-screenable, therefore there exists a σ-discrete closed refinement $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \underset{\sim}{V}_i$ of $\{W_x \mid x \in X\}$ where we denote $\underset{\sim}{V}_i = \{V_\alpha^i \mid \alpha \in \wedge_i\}$ for each i .

Let i be a fixed but an arbitrary integer. Let $V_\alpha^i \in \underset{\sim}{V}_i$ let us define

$$h(V_\alpha^i) = X - \cup(V_\beta^i \mid \beta \in \wedge_i \text{ and } \alpha \neq \beta) .$$

For j any integer let us define

$$\underset{\sim}{U}_{ij}(V_\alpha^i) = \{h(V_\alpha^i) \cap W \mid W \in \underset{\sim}{W}_j(x,V_\alpha^i)\}$$

where $\underset{\sim}{W}(x,V_\alpha^i) \in \{\underset{\sim}{W}_n(x)\}_{n=1}^{\infty}$ such that $V_\alpha^i \subset W_x$. Finally we define

$$\underset{\sim}{U}_{ij} = \{U : U \in \underset{\sim}{U}_{ij}(V_\alpha^i),\ \alpha \in \wedge_i\} \cup (X - U(V_\alpha^i \mid \alpha \in X_i)) .$$

Then $\underset{\sim}{U}_{ij}$ is an open cover of X for each i and j we now show

that $\{\underset{\sim}{U}_{ij}\}_{i,j=1}^{\infty}$ is a development of X.

Let $x_o \in X$ and let i_o be an integer such that $x_o \in V_{\alpha_o}^{i_o}$. Then if G is an open set containing x_o, then there is j_o such that $St(x_o, \underset{\sim}{W}_{j_o}(V_\alpha^i)) \subset G \cap W_x(V_\alpha^i)$ where $W_x(V_\alpha^i)$ is W_x such that $V_\alpha^i \subset W_x$. By construction x_o is not contained in any other element of $\underset{\sim}{U}_{i_o j_o}(V_\beta^i)$ for $\alpha \neq \beta$. Thus $St(x_o, \underset{\sim}{U}_{i_o j_o}) = St(x_o, \underset{\sim}{U}_{i_o j_o}(V_\alpha^i)) \subset St(x_o, \underset{\sim}{W}_{i_o}(V_\alpha^i)) \subset G \cap W_x(V_\alpha^i) \subset G$. Hence we have $St(x_o, \underset{\sim}{U}_{i_o j_o}) \subset G$. Hence the theorem is proved.

<u>Theorem 6.1.6</u> Let (X,d) be a conjugate strong quasi-metric space. Then, (X,τ_d) is a developable space.

<u>Proof</u>. Let (X,d) be quasi-metric space and $\tau_d \subset \tau_{d^1}$. Let us define $\underset{\sim}{B}_n = \{S(x,n) \mid x \in X\}$ where for each n and $x \in X$, $S(x,n) = \{y \mid d(x,y) < \frac{1}{2^n}\}$. We shall show that the countable family $\{\underset{\sim}{B}_n\}_{n=1}^{\infty}$ of open covers of X is such that $\{St(x,\underset{\sim}{B}_n)\}_{n=1}^{\infty}$ is a base for the neighborhood system with respect to τ_d.

Let $x \in X$ and U be an open set containing x. Then there is an n such that $x \in S(x,n-1) \subset U$. But $\tau_d \subset \tau_{d^1}$ implies that there is $m > n$ such that $x \in S^1(x,m) \subset S(x,n) \subset S(x,n-1) \subset U$, where $S^1(x,m) = \{y \mid d^1(x,y) < \frac{1}{2^m}\}$. We shall show that $St(x,\underset{\sim}{B}_m) \subset U$. Let $x \in S(y,m)$. Then $d(y,x) < \frac{1}{2^m}$, so that $y \in S^1(x,m) \leq S(x,n)$. Hence $S(y,m) \subset S(x,n-1) \subset U$, for if $z \in S(y,m)$ then we have $d(x,z) \leqq d(x,y) + d(y,z) \leqq \frac{1}{2^n} + \frac{1}{2^m} < \frac{1}{2^{n-1}}$,

i.e., $z \quad S(x,n-1)$. Hence the theorem is proved.

<u>Remark 6.1.1</u> It is not difficult to show that a metacompact developable T_3-space is conjugate strong quasi-metrizable.

CHAPTER VII

NAGATA SPACES AND METRIZABILITY OF SPACES

In 1961, Ceder [16] introduced a class of spaces called M_3-spaces. In 1966, Borges [13] named this class of spaces stratifiable and proved a number of theorems about stratifiable spaces. Ceder [16] showed that a topological space is Nagata iff it is first countable and an M_3-space.

In Section 1 we give an intrinsic characterization of Nagata spaces.

In Section 2 we give some mapping theorems.

In Section 3 we give some metrizability conditions which establish relations between various classes of spaces considered in this work and metric spaces. Theorem 7.3.3 is of special interest as it answers McAuley's question raised in [35].

<u>Definition 7.1.1</u> A <u>Nagata space</u> X is a T_1-space such that for each $x \in X$ there exist sequences of neighborhoods of x, $\{U_n(x)\}_{n=1}^{\infty}$ and $\{S_n(x)\}_{n=1}^{\infty}$ such that:

(1) for each $x \in X$, $\{U_n(x)\}_{n=1}^{\infty}$ is a base for the neighborhood system at x;

(2) for all $x,y \in X$, $S_n(x) \cap S_n(y) \neq \phi$ implies $x \in U_n(y)$.

Let $\underset{\sim}{P}$ be a collection of ordered pairs $P = (P_1, P_2)$ of subsets of X, with $P_1 \subset P_2$ for all $P \in \underset{\sim}{P}$. Then $\underset{\sim}{P}$ is called a <u>paired base</u> for X if P_1 is open for all $P \in \underset{\sim}{P}$ and if, for any $x \in X$ and any neighborhood U of x, there exists a $P \in \underset{\sim}{P}$ such that $x \in P_1 \subset P_2 \subset U$. Moreover, a collection $\underset{\sim}{P}$ of ordered pairs $P = (P_1, P_2)$ of subsets, with $P_1 \subset P_2$ for all $P \in \underset{\sim}{P}$ is called <u>cushioned</u> if for every $\underset{\sim}{P}^1 \subset \underset{\sim}{P}$, $\overline{\cup (P_1 \mid P \in \underset{\sim}{P}^1)} \subset \cup (P_2 \mid P \in \underset{\sim}{P}^1)$. $\underset{\sim}{P}$ is called <u>σ-cushioned</u> if it is the countable union of countably many cushioned subcollections.

<u>Definition 7.1.2</u> An <u>M_3-space</u> is a T_1-space with σ-cushioned paired base.

<u>Theorem 7.1.1</u> (J.G. Ceder [16]) A topological space X is a Nagata space iff it is a first countable and M_3-space.

<u>Definition 7.1.3</u> Let X be a topological space and $\underset{\sim}{P} = \bigcup_{i=1}^{\infty} \underset{\sim}{P}_i$, where $\underset{\sim}{P}_i = \{(P^i_{\alpha 1}, P^i_{\alpha 2}) \mid \alpha \in \Lambda_i\}$, for each i, is a σ-cushioned paired network. Then $\underset{\sim}{P}$ is a <u>σ-cushioned paired almost base</u> iff for $K \subset U$, where K is compact and U open in X, there is a finite subset F of $\bigcup_{i=1}^{\infty} \Lambda_i$ such that $K \subset \cup (P_{\alpha 1} \mid \alpha \in F) \subset \cup (P_{\alpha 2} \mid \alpha \in F) \subset U$.

<u>Definition 7.1.4</u> A sequence $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of covers (not necessarily open) of the space X is called <u>weakly majorized</u> by the open covering $\underset{\sim}{V}$ if for each point $x \in X$, and for some n, O_x (a neighborhood of x) and $V \in \underset{\sim}{V}$ we have $St(O_x, \underset{\sim}{V}_n) \subseteq V$.

A sequence $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ of covers (not necessarily open) of a space X is called weakly fundamental iff it is weakly majorized by every open cover of the space X and for each $x \in X$, $\{St(x,\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ is a base for the neighborhood system at x.

Definition 7.1.5 A sequence $\{\underset{\sim}{W}_i\}_{i=1}^{\infty}$ of covers (not necessarily open) of a space X is called a weak k-refining sequence iff for an arbitrary compact set $k \subset X$, $\{St(k,\underset{\sim}{W}_i)\}_{i=1}^{\infty}$ forms a base for the open sets which contain k.

Proposition 7.1.2 In a T_1-space X, the following statements are equivalent:

(a) X has a weakly fundamental sequence of covers;

(b) X has a weak k-refining sequence of covers.

Proof. (a) $\Longrightarrow$ (b) Let $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ be a weakly fundamental sequence of covers and not a weak k-refining sequence of covers. Without loss of generality let us assume that $\underset{\sim}{V}_{i+1}$ refines $\underset{\sim}{V}_i$ for each i. Now, since $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ is not a weak k-refining sequence, there exists a compact set k and an open set $U \supset k$ such that $St(k,\underset{\sim}{V}_i) \cap (X - U) \neq \phi$ for all i. Then, for each i there is x_i in k such that $St(x_i,\underset{\sim}{V}_i) \cap (X - U) \neq \phi$. k being compact, the sequence $\{x_i\}_{i=1}^{\infty}$ has a subsequence $\{x_{n_i}\}_{i=1}^{\infty}$ which converges to some point $x \in k$. Let us choose a neighborhood O_x^1 of x such that $St(O_x^1, \underset{\sim}{V}_{i(O_x^1,U)}) \subset U$. Then, obviously $St(x_{n_{i_o}},\underset{\sim}{V}_{n_{i_o}}) \subset U$ for some $n_{i_o} > i(O_x^1,U)$ as $x_{n_{i_o}} \in O_x^1$ for some i_o since $\{x_{n_i}\}_{i=1}^{\infty}$ converges to x. This contradicts the choice of the x_i's, completing

the first part of the proof.

(b) $\Longrightarrow$ (a). Suppose $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ is a weak k-refining sequence of covers of X, but not a weakly fundamental sequence of covers of X. Without loss of generality assume that $\underset{\sim}{V}_{i+1}$ refines $\underset{\sim}{V}_i$ for each i. Since $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ is not weakly fundamental, there exists a point $x \in X$ with a neighborhood 0_x such that for each neighborhood 0_x^1 of x we have for all i, $St(0_x^1, \underset{\sim}{V}_i) \cap (X - 0_x) \neq \phi$. In particular, this will be true if we put $0_{x,i}^1 = St(x, \underset{\sim}{V}_i)$. Let i_o be an integer such that $0_{x,i_o}^1 \subset 0_x$. This is possible as $\{St(x, \underset{\sim}{V}_i)\}_{i=1}^{\infty}$ is a base for the neighborhood system at x. Let us choose $x_i \in 0_{x,i}^1$ (for $i \geqq i_o$) for which $St(x_i, \underset{\sim}{V}_i) \cap (X - 0_x) \neq \phi$.

The sequence $\{x_i\}_{i \geqq i_o}^{\infty}$ converges to x as $\{St(x, \underset{\sim}{V}_i)\}_{i=1}^{\infty}$ forms a base for the neighborhood system at x. Obviously, $A = \{x_i\}_{i \geqq i_o}^{\infty} \cup \{x\}$ is a compact set. And for all $i \geqq i_o$ we have $St(A, \underset{\sim}{V}_i) \cap (X - 0_x) \supset St(x_i, \underset{\sim}{V}_i) \cap (X - 0_x) \neq \phi$ which contradicts the fact that $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ is a weak k-refining sequence of covers of X.

<u>Theorem 7.1.3</u> For a T_1-space the following statments are equivalent:

(a) X is a Nagata space;

(b) X has a weakly fundamental sequence of covers;

(c) X has a weak k-refining sequence of covers.

<u>Proof</u>. By Proposition 7.1.2 (b) $\Longleftrightarrow$ (c).

(b) $\Longrightarrow$ (a) Let $\{\underset{\sim}{V}_i\}_{i=1}^{\infty}$ be a weakly fundamental sequence

of covers of X. It is easy to see that for each $x \in X$, $\{St(St(x,\underset{\sim}{V}_i),\underset{\sim}{V}_i)\}_{i=1}^{\infty}$ is a base for the neighborhood system at x. Now, for each i define $U_i(x) = St(St(x,\underset{\sim}{V}_i),\underset{\sim}{V}_i)$ and $S_i(x) = St(x,\underset{\sim}{V}_i)$. Suppose $S_i(x) \cap S_i(y) \neq \phi$, then, clearly $y \in U_i(x)$ for each i. Thus, X is a Nagata space.

(a) $\Longrightarrow$ (b) Let X be a Nagata space. Then by definition there exist sequences $\{U_n(x)\}_{n=1}^{\infty}$ and $\{S_n(x)\}_{n=1}^{\infty}$ of neighborhoods of x such that

(1) for each $x \in X$, $\{U_n(x)\}_{n=1}^{\infty}$ is a local base of neighborhoods of x;

(2) for all $x,y \in X$, $S_n(x) \cap S_n(y) \neq \phi$ implies $x \in U_n(y)$.

Without loss of generality let us assume that $S_n(x) \subset U_n(x)$, $S_{n+1}(x) \subset S_n(x)$, $U_{n+1}(x) \subset U_n(x)$ for each n and $x \in X$. Now, define $d : X \times X \to R$ by setting $d(x,y) = \frac{1}{n}$, where n is the largest integer for which $y \in S_n(x)$ and $x \in S_n(y)$. It is not difficult to show that d is a semi-metric for X. For each positive integer n, let $\underset{\sim}{V}_n$ be the collection of all sets of diameter less than $\frac{1}{n}$. Then for each n, $S(x; \frac{1}{n}) = St(x,\underset{\sim}{V}_n)$, for let $y \in S(x; \frac{1}{n})$. Then $A = \{x,y\} \in \underset{\sim}{V}_n$ implies $y \in St(x,\underset{\sim}{V}_n)$. On the otherhand, let $y \in St(x,\underset{\sim}{V}_n)$. Then there is $A \in \underset{\sim}{V}_n$ such that $x,y \in A$, and therefore, $d(x,y) \leq$ diameter of $A < \frac{1}{n}$. Thus $y \in S(x, \frac{1}{n})$, and $\{St(x,\underset{\sim}{V}_n)\}_{n=1}^{\infty}$ is a base for the neighborhood system at x.

It remains to show that $\{St(St(x,\underset{\sim}{V}_n),\underset{\sim}{V}_n)\}_{n=1}^{\infty}$ is a base for the neighborhood system at x. Let $x \in X$ and U be any neighborhood of x. Suppose for all n we have $St(St(x,\underset{\sim}{V}_n),\underset{\sim}{V}_n) - U \neq \phi$, i.e., for each n there is $x_n \in St(St(x,\underset{\sim}{V}_n),\underset{\sim}{V}_n) - U$. For each n there is A_n and B_n in V_n such that $x_n \in A_n$, $x \in B_n$ and $A_n \cap B_n \neq \phi$. Now, by the definition of d, we have $S_n(x) \cap S_n(x_n) \neq \phi$ for each n. Hence, for some m, $x_m \in U$ for which $U_m(x) \subset U$, which contradicts the construction of the sequence $\{x_m\}$. Hence the theorem is proved.

<u>Theorem 7.1.4</u> A regular space is Nagata iff it is first countable and has σ-cushioned paired almost base.

<u>Proof</u>. Necessity follows trivially. Sufficiency follows on the same line as Theorem 4.2.3 in [46].

<u>Corollary 7.1.4A</u> Every Nagata space is semi-metric.

<u>Definition 7.2.1</u> A function $f : X \to Y$ is called <u>σ-cushioned paired mapping</u> iff f is such that for any σ-cushioned paired open base $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \{(V_{\alpha 1}^i, V_{\alpha 2}^i) \mid \alpha \in \wedge_i\}$ the $f(\underset{\sim}{V}) = \bigcup_{i=1}^{\infty} \{(f(V_{\alpha 1}^i), f(V_{\alpha 2}^i)) \mid \alpha \in \wedge_i\}$ is a σ-cushioned paired cover of Y which need not be open or closed.

<u>Theorem 7.2.1</u> For a T_1-space Y the following statements are equivalent:

(1) Y has a σ-cushioned paired network;

(2) Y is the image of a M_3-space under a σ-cushioned paired mapping which is one to one and continuous.

Proof. (1) ==> (2) Let $\underset{\sim}{V}$ be a σ-cushioned paired network for Y . Then there exists a σ-cushioned paired network $\underset{\sim}{U}$ as in Proposition 2.1.4. Let X be a copy of Y topologized by taking $\underset{\sim}{U}$ as base for the topology of X . Then, obviously X is a T_1-space with σ-cushioned paired base, i.e., X is an M_3-space. Let $f : X \to Y$ be the identity transformation. Clearly f is one to one continuous. f is σ-cushioned follows by the construction of base for X .

Remark 7.2.1 It is to be noted that σ-cushioned paired mapping need not be closed or open. This follows since there exist non M_3-spaces which have σ-cushioned networks. Hence, if σ-cushioned mappings implied closedness or openness of f by the above theorem, every T_1-space with σ-cushioned paired network would be an M_3-space which is not true.

Definition 7.2.2 A regular space X is called an M_1-space iff it has σ-closure preserving open base.

Definition 7.2.3 A function $f : X \to Y$ is called σ-closure preserving mapping iff for any σ-closure preserving open base $\underset{\sim}{V} = \bigcup_{i=1}^{\infty} \{V^i \mid \alpha \in \wedge_i\}$ the collection $f(\underset{\sim}{V}) = \bigcup_{i=1}^{\infty} \{fV^i \mid \alpha \in \wedge_i\}$ is a σ-closure preserving cover of X .

Theorem 7.2.2 For a regular space Y, the following statements are equivalent:

(1) Y has σ-closure preserving network;

(2) Y is an image of an M_1-space under a σ-closure preserving mapping which is one to one and continuous.

Proof. Is similar to Theorem 7.2.1.

Remark 7.2.2 There exist regular spaces with σ-closure preserving networks which are not M_3-spaces and so not M_1-spaces by Ceder [16]: following Remark 7.2.1 we can now show that a σ-closure preserving mapping need not be closed or open.

Remark 7.2.3 We feel that Theorem 7.2.1 and Theorem 7.2.2 will probably help in answering the question of Ceder [16]: "Is an M_3-space an M_1-space"?

Theorem 7.3.1 (Tamano [55]) A completely regular space X is metrizable iff $X \times BX$ is normal and $\Delta = \{(x,x) \mid x \in X\}$ is a closed G_δ-set in $X \times BX$, where BX is any Hausdorff compactification of X.

Theorem 7.3.2 If X is a completely regular space, then X is metrizable iff X is a p-space with G_δ-diagonal and $X \times BX$ is normal, where BX is a Hausdorff compactification of X.

Proof. Follows from Theorems 6.1.1 and 8.1.1.

Remark 7.3.1 In view of Theorem 8.1.1, it is easy to see that the normal Moore space problem (is every normal Moore space metrizable?) is equivalent to asking whether or not product of normal Moore space with any of its Hausdorff compactification is normal.

The following theorem gives a factorization of a metrization Theorem 4 by Bing [11] and answers the question raised by McAuley [35].

Theorem 7.3.3 A completely regular space X is metrizable iff the following conditions are satisfied:

(a) X has σ-cushioned paired network;

(b) X is first countable;

(c) X is a p-space;

(d) X is collectionwise normal.

Proof. By Theorem 4.1.4 (a) and (b) imply that X is a semi-metric space. By Theorem 6.1.5 (a), (b) and (c) imply X is developable. Finally by Bing [11] (a), (b), (c) and (d) combined together imply X is metrizable. The converse is well-known.

Remark 7.3.2 For the metrizability of X in Theorem 7.3.3, condition (b) is redundent as (a) and (c) implies (b) by Theorem 6.1.4.

Theorem 7.3.4 A Nagata space X is metrizable iff any one of the following conditions are satisfied:

(i) X is a p-space;

(ii) X has point countable base;

(iii) X is quasi-metrizable.

Proof. (i) If a Nagata space is a p-space by Corollary 7.1.4A and Corollary 6.1.4A, it is a developable. Since Nagata space is also paracompact by Theorem 2.2 of Ceder [16]. By Bing [11] X is metrizable.

(ii) See Theorem 2 of Heath [25].

(iii) If X is a Nagata space which is also quasi-metrizable. By Theorem 5.2.2 it is easy to show that there exist sequences $\{U_n(x)\}_{n=1}^{\infty}$ and $\{S_n(x)\}_{n=1}^{\infty}$ of neighborhoods satisfying

(a) $\{U_n(x)\}_{n=1}^{\infty}$ is a base for the neighborhood system at x,

(b) for all $x,y \in X$, $S_n(x) \cap S_n(y) \neq \phi$ implies $x \in U_n(y)$.

(c) for all $x,y \in X$, $y \in S_n(x)$ implies $S_n(y) \subset U_n(x)$.

Hence by Theorem 1 of Nagata [44] X is metrizable. The converse is obvious.

Theorem 7.3.5 A regular σ-paracompact M-space X is metrizable iff X has G_δ-diagonal.

Proof. By Theorem 6.1 of [41], there exists a metrizable space Y and a mapping f from X onto Y such that f is continuous closed, and $f^{-1}y$ is countably compact for each $y \in Y$. Thus

by our Proposition 3.3.11A, $f^{-1}y$ is compact, and so f is a perfect mapping. Thus, by Arhangel'skii [4] X is a paracompact p-space with G_δ-diagonal. Hence, by Corollary 6.1.2A and Bing [11], X is metrizable.

Corollary 7.3.5A A regular semi-metric space is metrizable iff it is an M-space.

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